VECTOR MISCONCEPTIONS IN FINNISH MATRICULATION EXAMINATION

Heli Virtanen, Anne-Maria Ernvall-Hytönen and Antti Laaksonen

University of Helsinki

ABSTRACT

We used a large data set to discover what kind of misconceptions Finnish secondary school students have in elementary vector calculations. The study was based on a data set from the spring 2020 advanced mathematics exam in the Finnish matriculation examination. The data set consisted of answers from 13,284 students who participated in the exam. We focused on a question about vectors and compared the results to another question about equations and inequalities. We analyzed and classified the answers and explain common misconceptions the students had in the exam. Certain typical mistakes included not understanding that the result of a dot product is a scalar, and not a vector, or not understanding the notation for the norm of a vector.

INTRODUCTION

Vectors are a widely used concept in mathematics and physics. Typically, students first encounter vectors during their secondary education. In previous studies, secondary school students were reported to have various difficulties with vectors (Jewaru et al., 2021; Tairab et al., 2020). However, sample sizes in those studies were small.

The aim of our study is to discover what kind of misconceptions regarding basic vector operations can be found from a large data set of students' answers. We used a data set from the Finnish matriculation examination which is taken at the end of the Finnish upper secondary school. We focused on the spring 2020 advanced mathematics exam which contained an exercise on vectors. The number of students who participated in the exam was 13,284.

BACKGROUND AND PREVIOUS RESEARCH

Interviews and tests have been used to study how well secondary school students perform vector calculations. Difficult topics with vector calculations include direction vector, vector magnitude, adding vectors and multiplying vectors by numbers (Jewaru et al., 2021; Tairab et al., 2020).

VERTAISARVIOITU
KOLLEGIALT GRANSKAD
PEER-REVIEWED
www.tsv.fi/tunnus

Vector misconceptions have been studied in a more physics related setting than a pure mathematics setting. However, since the basic vector calculations at high school level are those widely used in introductory university physics courses, and graduating high-school students will become new university students, we can also see what kind of misconceptions occur during these courses. In a recent study, the students' understanding of vector concepts was still found to be lacking, especially in graphical representations (Latifa et al., 2021).

Barniol and Zavala (2014) developed a test of understanding vectors (TUV) by first observing what kind of errors university students frequently make and then creating a multiple-choice questionnaire based on the errors. They focused on the following ten introductory physics course concepts: direction of vector, magnitude of vector, component of vector, unit vector, graphical representation, vector addition, vector subtraction, scalar multiplication, dot product, and cross product. Except for cross product, these concepts are part of the Finnish upper secondary school curriculum.

A test on scalar multiplication (Barniol & Zavala, 2012) revealed that it is easier to multiply a vector by a positive scalar than a negative scalar. The major four different error categories using a graphical approach are 1) incorrect magnitude with correct direction; 2) perpendicular (in either clockwise or counterclockwise) direction with correct magnitude; 3) translating the original vector; and 4) opposite direction with correct magnitude.

Similarly, subtraction of vectors was found to be significantly more difficult than addition, and dot product and vector direction were also more difficult than average (Susac et al., 2018).

The difficulties high school students face with vectors have been studied by Harel (1990) and more recently, by Demetriadou and Tzanakis (2010) in a more mathematical setting. In the case of university students, Sandoval and Possani (2016) and Stewart and Thomas (2009) studied extensively vector misconceptions and linear algebra. However, these cases focused on more complicated situations than were possible with our data.

DATA, METHODS, AND ANALYSIS

We investigated students' answers to two questions in the spring 2020 advanced mathematics exam in the Finnish matriculation examination. The first question was about equations and inequalities. The second question was about vectors. For each student (n=13,284), our data set contained their answers to the questions and the awarded points.

Research questions

Our intention was to investigate the following questions:

1. How good are the Finnish secondary school students with basic properties of vectors? Furthermore, if we look at students with varying skill levels (based on a very basic exam question), are there differences between these groups?
2. What kind of misconceptions occur, and how common are these?
3. What kind of misconceptions or mistakes cluster together?

Exam setting

The students answered the questions using a computer. While sitting the exam, they were allowed a standardized formula book, scratch paper, and writing utensils. Students were monitored throughout the entire exam.

The exam consisted of 13 questions, divided into the following three sections A (4 questions, all mandatory), B1 (5 questions, choose 3) and B2 (4 questions, choose 3). In sections B1 and B2, students were allowed calculators and computer programs capable of symbolic computations. In total, students had 6 hours for the entire exam, but they had to return part A before getting the symbolic calculators allowed only in parts B1 and B2. In part A, students were allowed to have scientific calculators not capable of symbolic calculations.

Questions

We studied the first two questions of the exam, which were in section A. Both the questions consist of several small tasks. Unlike in typical mathematical problems, the students were asked to only give the answer for each task with no reasoning behind the answer.

The maximum number of points for each question was 12; the available points are in parentheses for each task. The exam was held in both Finnish and Swedish, and the data included students of both language groups. The same questions were given to both groups.

Question 1

1.1. Solve the equation $-4x + 2 = 0$. (2 p)
1.2. Solve the inequality $x + 4 < -6$. (3 p)
1.3. Solve the equation $x^6 + x^3 = 0$. (3 p)
1.4. Which numbers $x \in \mathbb{R}$ satisfy both inequalities $-3x + 6 < 0$ and $x^2 - 9 < 0$? (4 p)

Question 2
Let us look at the vectors $\bar{\boldsymbol{a}} = \mathbf{7}\bar{\boldsymbol{\imath}} + \mathbf{2}\bar{\boldsymbol{\jmath}}$ and $\bar{\boldsymbol{b}} = -\mathbf{3}\bar{\boldsymbol{\imath}} + \mathbf{5}\bar{\boldsymbol{\jmath}}$.
2.1. Compute $\bar{\boldsymbol{a}} + \bar{\boldsymbol{b}}$. (2 p)
2.2. Compute $\mathbf{2}\bar{\boldsymbol{b}} - \bar{\boldsymbol{a}}$. (2 p)
2.3. Compute $|\bar{\boldsymbol{b}}|^{\mathbf{2}}$. (2 p)
2.4. Compute the length of the vector $\bar{\boldsymbol{a}} + \bar{\boldsymbol{b}}$ with the accuracy of two decimals. (2 p)
2.5. Compute $\bar{\boldsymbol{a}} \cdot \bar{\boldsymbol{b}}$. (2 p)
2.6. Compute the angle between vectors $\bar{\boldsymbol{a}}$ and $\bar{\boldsymbol{b}}$ with the accuracy of one degree. (2 p)

In all these tasks, full score (2 or 3 points, depending on the task) was awarded for a perfect answer. Partial score was awarded for answers which contained only minor errors or extra elements in addition to the correct answer. For example, in task 1.3 partial score was awarded for students who only found one root. The scoring will be discussed in more detail later.

Methods

We examined the answers both quantitatively and qualitatively. The underlying idea in looking at both questions was that we wanted to have some kind of understanding about the mathematical level of the students. Question 1 measured very basic skills in mathematics. Therefore, we thought that it would allow us to separate between weaker and stronger students. We calculated the mode, median, and mean of points for each question, and the Pearson correlation between the points received from question 1 and question 2 to discover dependencies between the questions.

We analyzed more thoroughly question 2 answers that were not awarded full points. First, the answers were standardized by removing any text other than the answer, including extra spaces. Then, we calculated how many of each of these answers there were and ranked them in the order of most common to least common. Finally, we grouped the answers into categories based on what kind of errors or assumed misconception was present in the answer. We analyzed the groups with the largest number of students to identify the most common errors and misconceptions. We also created a code profile for tasks 2.3 to 2.6 of the top 20 most common answers to examine if one error meant another error in another task.

Analysis

The first question was used to get a preliminary idea about the students' mathematical skills. In general, the students did well in that question. In 1.1, 12,649/13,284 students and in 1.2, 11,797/13,284 students got the full score. The distribution in tasks 1.3 and 1.4 was a bit more varying. These are illustrated in Figure 1 for task 1.3 and in Figure 2 for task 1.4. Essentially, the students were capable of solving first degree equations and inequalities, but higher degree tasks caused more problems, even though in average, these tasks went well.

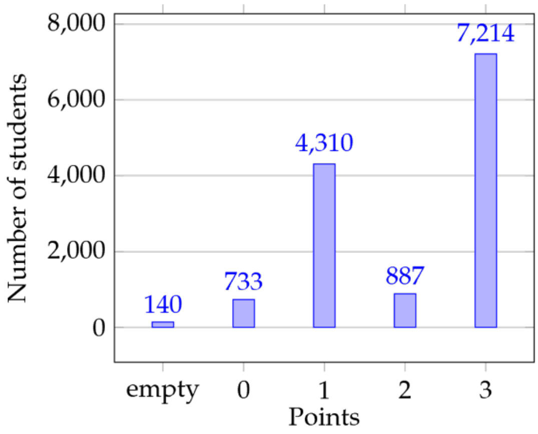

Figure 1. Point distribution for task 1.3 (solving equation $x^6 + x^3 = 0$).

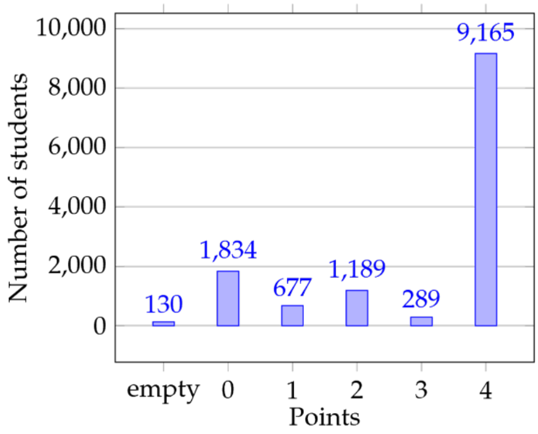

Figure 2. Point distribution for task 1.4 (solving inequalities $-3x + 6 < 0$ and $x^2 - 9 < 0$).

The mode of points in both questions was 12 (full points), while the median was 10. The mean was 9.9 in question 1, and 9.1 in question 2. The Pearson correlation was 0.471 between the points in question 1 and question 2; this correlation was significant at the 0.01 level. We also calculated the Pearson correlations between different tasks. These correlations were generally weak but statistically significant. Figure 3 provides an overview of the results between these two questions. We have cross tabulated the point distributions between question 1 and question 2 and coloured them so that darker violet indicates a larger population of students. Figure 3 shows that in general, those who did poorly in question 1, also did poorly in question 2. However, also many of those who did well in question 1, did poorly in question 2. Therefore question 1 does indeed give a rough initial idea of the mathematical skills of the students. To better describe how different student populations are distributed, we computed the percentage distribution of scores in question 1 for each possible score in question 2 in Figure 4.

To further analyze the dependence of the questions, we divided the students into four groups based on their score in question 1. We tried to get as equivalently sized groups as possible, but an equal division was not possible due to the large proportion of students who had a good score. We created the following groups: students with 12 points (5,381 students, corresponding to 41.3% of the population), students with 10 or 11 points (3,637/27.9%), students with 8 or 9 points (1,923/14.5%) and finally, students with at most 7 points (2,089/15.7%). We then compared the performance of these groups in tasks 2.1–2.6 using a Kruskal-Wallis test on these groups. The mean ranks are listed in Table 1. In general, we can see that in task 2.1, the difference between the weakest and strongest students is not large, but it is considerably larger in later tasks.

q1\q2	0	1	2	3	4	5	6	7	8	9	10	11	12
0	2	0	0	2	2	0	0	0	0	0	0	0	0
1	0	0	2	2	2	0	2	0	1	0	1	0	0
2	2	1	5	8	13	4	7	2	3	0	2	0	1
3	2	2	10	21	24	7	17	5	9	2	5	1	2
4	1	1	12	17	44	16	18	14	18	4	10	4	3
5	3	3	12	29	65	22	42	19	31	17	21	10	10
6	3	2	17	19	95	40	104	48	96	45	89	42	62
7	0	2	17	32	64	36	73	34	81	25	76	32	43
8	3	2	11	29	78	41	108	61	130	71	206	80	148
9	0	1	12	26	64	41	84	44	134	54	148	48	113
10	4	2	22	52	168	89	289	117	389	169	538	237	620
11	0	0	3	11	40	13	46	40	98	55	146	67	166
12	2	0	5	20	66	55	201	131	444	240	1113	699	2244

Figure 3. Total score from question 1 versus question 2. Number of points from question 1 in vertical axis and number of points from question 2 in horizontal axis.

q1\q2	0	1	2	3	4	5	6	7	8	9	10	11	12	Total
0	9 %			1 %	0 %									0 %
1			2 %	1 %	0 %		0 %		0 %		0 %			0 %
2	9 %	6 %	4 %	3 %	2 %	1 %	1 %	0 %	0 %		0 %		0 %	0 %
3	9 %	13 %	8 %	8 %	3 %	2 %	2 %	1 %	1 %	0 %	0 %	0 %	0 %	1 %
4	5 %	6 %	9 %	6 %	6 %	4 %	2 %	3 %	1 %	1 %	0 %	0 %	0 %	1 %
5	14 %	19 %	9 %	11 %	9 %	6 %	4 %	4 %	2 %	3 %	1 %	1 %	0 %	2 %
6	14 %	13 %	13 %	7 %	13 %	11 %	11 %	9 %	7 %	7 %	4 %	3 %	2 %	6 %
7		13 %	13 %	12 %	9 %	10 %	7 %	7 %	6 %	4 %	3 %	3 %	1 %	4 %
8	14 %	13 %	9 %	11 %	11 %	11 %	11 %	12 %	9 %	10 %	9 %	7 %	4 %	8 %
9		6 %	9 %	10 %	9 %	11 %	9 %	9 %	9 %	8 %	6 %	4 %	3 %	6 %
10	18 %	13 %	17 %	19 %	23 %	25 %	29 %	23 %	27 %	25 %	23 %	19 %	18 %	22 %
11			2 %	4 %	6 %	4 %	5 %	8 %	7 %	8 %	6 %	6 %	5 %	6 %
12	9 %		4 %	8 %	9 %	15 %	20 %	25 %	31 %	35 %	47 %	57 %	66 %	43 %
Total	100 %	100 %	100 %	100 %	100 %	100 %	100 %	100 %	100 %	100 %	100 %	100 %	100 %	100 %

Figure 4. The proportions of student scores in question 1 normalized against scores in question 2. Number of points from question 1 in vertical axis and number of points from question 2 in horizontal axis.

Table 1. Mean ranks in tasks 2.1–2.6 for different student groups based on question 1.

q1 points	2.1	2.2	2.3	2.4	2.5	2.6
0-7	6,256.83	5,417.83	4,766.17	4,638.30	4,700.25	4,217.87
8, 9	6,473.19	6,164.23	5,907.98	5,764.85	5,885.69	5,272.18
10, 11	6,532.14	6,538.89	6,365.33	6,385.70	6,270.95	5,791.19
12	6,618.60	7,042.62	7,454.35	7,282.86	7,468.41	7,286.43

Figure 5 shows the point distribution for question 2. Generally, the tasks seem to be in increasing difficulty order, except for task 2.3. In this task the students were asked to evaluate a formula, and it was more difficult than task 2.4 about the length of a vector. The angle between two vectors (task 2.6) was the most difficult task. In that task, only 36.5% of the students got the correct answer.

Some misconceptions can be seen from the guidelines in assigning points (Ylioppilastutkintolautakunta, 2020). Question 2 consisted of six tasks, all graded on a scale of 0, 1 or 2 points. For all the tasks in question 2, if a student provided an excessively long answer (over 30 characters), they were given at most 1 point instead of the normal maximum of 2 points for the correct answer. The answers also only accepted text, with instructions to write the vectors $\bar{\boldsymbol{\imath}}$ and $\bar{\boldsymbol{\jmath}}$ as i and j, respectively.

For task 2.1, students could receive 1 point if they got one of the coefficients correct or had some wrong notations in their answer. For task 2.2, 1 point was awarded if a student had one correct coefficient, mixed up i and j, wrote both vectors as either i or j, or had some other wrong notations in their answer. For task 2.3, an incorrect accuracy of the answer gave 1 point, and the answers $5.83 \approx |\bar{\boldsymbol{b}}|$ and $53 = |\bar{\boldsymbol{a}}|^2$ were also given 1 point. For task 2.4, 1 point was given if the answer was one of the following: 8.1, 8.062, -8.06, or ± 8.06. If the answer was given at some other accuracy or there was a rounding error, students did not receive any points. In task 2.5, a sign error in the answer gave 1 point; if the answer contained vectors, the student got 0 points. In task 2.6, 1 point was given for an incorrect accuracy with correct rounding and the answers 2, 1.8 or 1.83, where the calculator was set to radians instead of degrees. If a student had the answer 75, which can be obtained with a sign error on the dot product $\bar{\boldsymbol{a}} \cdot \bar{\boldsymbol{b}} = \mathbf{11}$, they were given 1 point. (Ylioppilastutkintolautakunta, 2020)

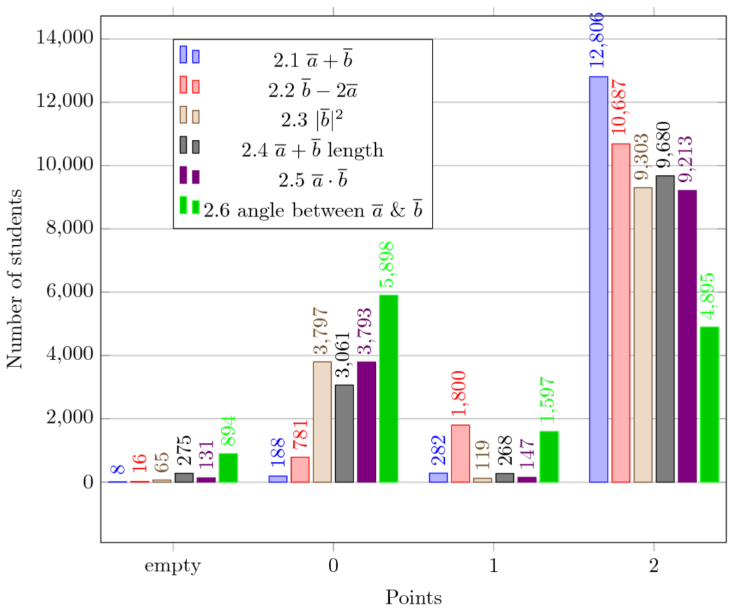

Figure 5. Question 2 point distributions for all six tasks.

RESULTS

Overall performance

Our premise was to use question 1 to measure the overall performance of the students and to determine if basic skills in equation solving had consequences on performance in vectors. Based on the mode (12 in both questions), median (10 in both questions), and the means (9.9 for question 1 and 9.1 for question 2), the students performed very well. Based on the data from the Kruskal-Wallis test and the Pearson correlation, there is a dependence between these questions. However, the dependence is not that straightforward. Looking at Figures 6 and 7, those who did well in question 1 did not necessarily do well in question 2. For instance, if we consider the students with 10 points in question 1, their point distribution in question 2 is quite like the point distribution of students in general, with a peak at 6 points. This suggests that one could do relatively well in question 1 even without having the skills needed to do well in question 2. Doing well in question 2 suggests that one probably also did well in question 1. In addition, it seems that those who solved question 1 with full 12 points, also did well in question 2.

As the students were instructed to write the vectors $\bar{\boldsymbol{\imath}}$ and $\bar{\boldsymbol{\jmath}}$ as i and j, we will also use the same style when discussing the student answers.

Tasks 2.1 and 2.2: computing sum of vectors and multiplying by a negative scalar

Most students had correct answers in tasks 2.1 and 2.2. In task 2.1, 12,806 (96.4%) students had the correct answer 4i+7j, and in task 2.2, 10,687 (80.45%) students had the correct answer -17i+j. The answer categories for tasks 2.1 and 2.2 and the number of students for each category can be found in

Table 2. The most common type of error in both these two tasks had to do with a sign of a component, with 202 (1.5%) students in task 2.1 and 1,563 (11.76%) students in task 2.2. In task 2.1, these errors appeared for example in the following answers: 10i+7j (-3i to 3i, 31 students), 4i-3j (5j to -5j, 26 students), -4i+7j (added sign in front of 4i, 24 students), -10i+3j ($-\bar{a} + \bar{b}$, 56 students) or 10i-3j ($\bar{a} - \bar{b}$, 27 students). In task 2.2, sign errors appear in answers -17i+9j ($+2 \cdot 2\bar{\jmath}$, 388 students), -17i-j (j to -j, 381 students), 17i+j (-17i to 17i, 298 students), 11i+9j (b+2a, 217 students), 11i+j (-14i to 14i, 89 students), 17i-j (-(b-2a), 76 students), with -3i to 3i and 5j to -5j appearing as well. In task 2.1, the second most common category for incorrect answers are what we consider minor errors (99 students in total). These errors include typographical errors, such as writing the vector i as j or vice versa (for example, 4j+7j, 6 students; or 4i+7i, 9 students) or possible calculation errors (7-3=5), including accidentally pressing the key next to the correct key (for example, 5i+7j, 57 students, pressing the key 5 instead of the key 4 or miscalculation) in this category. As the data only included the answers, we cannot know

what the student actually calculated. These kinds of errors also appeared in task 2.2, with 56 students.

For task 2.2, the second most common category with errors (149 students) was dropping out a component: -14i+j (-3i has dropped, 48 students), -17i+3j (-1 times 2j, 75 students), and 17i+5j (2j dropped, 11 students). This was also present in the first task for 40 students, which was the fourth most common error category.

An interesting error type for both tasks 2.1 and 2.2 was an answer that is a number, for example 11 (task 1, 38 students, can be computed for example with $7 + 2 - 3 + 5$, meaning the vectors were ignored) or -16 (task 2, 24 students, computed $-17 + 1$, again, vectors were ignored). In total, 40 students had a numerical answer in task 1 and 55 students in task 2. Of the 53 students who got a scalar for the first task, 51 students got a scalar in the second task, one got the second one correct and one left the answer empty. Out of these 53 students, 42 did get at least one correct answer in the other 5 tasks.

Task 2.2 also had the error of an excessively long answer (43 students), where the students had the correct answer, but they had written something on how they got there, making the answers longer than 30 characters and thus not being in the correct category for 2 points. Another error worth mentioning was switching the vectors vice versa (32 students).

Table 2. Question 2, task 1 and 2 categories for the students' answers.

Task 1 category	Number of students (T1)	Task 2 category	Number of students (T2)
Correct (4i+7j)	12,806 (96.4%)	Correct (-17i+j)	10,687 (80.45%)
Sign error	202 (1.5%)	Sign error	1,563 (11.76%)
Minor error	99 (0.74%)	Dropped a component	149 (1.12%)
Answer is a number	53 (0.39%)	Answer is a number	59 (0.41%)
Dropped a component	40 (0.3%)	Minor error	56 (0.42%)
Other	76 (0.57%)	Other	754 (5.67%)
Empty	8 (0.06%)	Empty	16 (0.12%)

Tasks 2.3, 2.4 and 2.5: Length of a vector and dot product

Even though the success rate was lower than for the first two tasks, tasks 2.3, 2.4 and 2.5 were still solved correctly by 9,303 (70.03%), 9,680 (72.86%) and 9,213 (69.35%) students, respectively. The categories for these tasks are shown in

Table 3 for task 2.3, in Table 4 for task 2.4, and in Table 5 for task 2.5.

The most common error category for tasks 2.3 and 2.4 is that the result is a vector. For task 2.3, the correct answer can be calculated using the formula $|\bar{\mathrm{b}}|^2 = \left(\sqrt{(-3)^2+5^2}\right)^2 = (-3)^2 + 5^2 = 9 + 25 = 34$. However, a significant number of students obtained the result 9i+25j or a variation of it, with different signs in front of the terms, or in few cases even multiple vectors. All the answers with some variation of 9i+25j were categorized to “9i+25j”, and this category had 1,930 students (14.52%). Similarly, for task 2.5, the correct calculation would have been $7 \cdot (-3) + 2 \cdot 5 = -21 + 10 = -11$. In total 2,781 (20.93%) students were in category “21i+10j” which means their answer was -21i+10j or 21i+10j or some variation thereof. Of the 1,930 students in category “9i+25j” for task 2.3, 1,191 students were also in the category “21i+10j” for task 2.5. Other vector-like answers were given by 511 (3.84%) students in task 2.3 and 167 (1.25%) students in task 2.5.

The second most common type of error in tasks 2.3 and 2.5 was using the vectors i and j as variables which means that the answer for task 2.3 was for example 9i^2-30ij+25j^2 or 9i^2-15ij-15ji+25j^2. In total, 708 (5.32%) students belonged to these categories in task 2.3. Similarly, in 2.5, variable-like behavior was found for 276 (2.07%) students. A total of 145 students belong to both these categories.

In task 2.3, 98 students (0.7%) seemingly knew what they were doing, but rounded in a middle step. The students with other numerical values or other answers were gathered in the category “other”, which had 767 (5.77%) students.

One interesting observation concerning task 2.3 is the following: if we compare the students having a scalar as the answer with the students not having a scalar, the students not having a scalar did poorer in other problems than the students in general. For example, their most common answer to task 1.3 was to find just one solution (while most of the students found both solutions).

In task 2.5, 139 (1.04%) students had the answer 31, which can be the result of forgetting the sign of -3i and calculating with just 3i to obtain the result 21+10=31. Similarly, 130 (0.97%) students ended up with the result 11, which could be done by removing the sign in front of a correctly calculated -11, or by just removing the vectors i and j and adding the remaining numbers together. Some students with the correct answer of –11 also had incorrect reasoning behind it. Namely, some of them had written how they obtained the answer: explanations include -21+10j=-11, where the j might have been a typographical error, or stating that the vectors do not matter. A total of 264 (1.98%) students obtained some other positive number and 130 (0.97%) students obtained a negative number.

In task 2.4, the correct answer with accuracy of two decimals was 8.06, and 41 (0.3%) students had the correct answer but gave the answer with one or three decimals. Other results that were approximately 8 were given by 407 (3.06%) students; 25 (0.18%) students also gave the correct answer of sqrt(65) but did not calculate the final answer. A total of 29 (0.21%) students forgot to calculate the square root, yielding the answer of 65. All the answers discussed this far were somewhat correct but missing something: accuracy, the length of the answer, not calculating the final answer, or forgetting to take the square root. However, more interestingly, 795 (5.98%) students had the answer of 13.11 which means they most likely computed $|\bar{a}+\bar{b}| = |\bar{a}| + |\bar{b}| = \sqrt{53} + \sqrt{34} = 13.11$. A variant of this error is $|\bar{a}+\bar{b}| = \sqrt{|\bar{a}|^2 + |\bar{b}|^2} = \sqrt{53+34} = 9.32$, given by 119 (0.89%) students. The number 11 shows up in 216 (1.62%) answers which means that the students either calculated the correct vector 4i+7j and removed the vectors, or summed all the components together (like in task 2.1) or calculated the dot product and removed the sign. A total of 78 (0.58%) students had the answer $7.28 \approx |\bar{a}|$.

The task 2.4 also had vector answers (177, 1.33% students) and variable-like answers (19 students). However, in this task, these categories were not as common as in tasks 2.3 or 2.5.

Task 2.6: computing the angle between the vectors

This was the most difficult task in our data, with 4,895 (36.84%) correct answers. The grading guidelines (Ylioppilastutkintolautakunta, 2020) give indications on the most common errors. These were an incorrect sign of the dot product resulting in the answer 75 with 674 (5.07%) students, and the answers around 2, meaning the calculator was set to radians, by 597 (4.49%) students. Incorrect accuracy of not rounding the result to an integer degree was seen in 175 (1.31%) answers. The answer angle of 1 was seen in 519 (3.90%) answers, angle 26 in 237 (1.78%) answers, angle 90 in 230 (1.7%) answers, and angle 100 in 199 (1.49%) answers. Other positive angles were given in 4,698 (35.36%) answers. Some students answered with a formula or part of the formula; the answers by 48 (0.36%) and 15 (0.11%) students indicated there either was not an angle between the vectors or the angle could not be negative, respectively. There were some students that also obtained a number like 0.26, but answered 26 instead, or had a percentage sign at the end of their answer. The categories are shown in Table 6.

Table 3. Answer categories for task 2.3

Task 2.3 categories	Number of students
Correct (34)	9,303 (70.03%)
“9i+25j”	1,930 (14.52%)
Variable-like	708 (5.32%)
Other vector-like	511 (3.84%)
Other	767 (5.77%)
Empty	65 (0.49%)

Table 4. Answer categories for task 2.4.

Task 2.4 categories	Number of students
Correct (8.06)	9,680 (72.86%)
$13.11 \approx \lvert \bar{a} \rvert + \lvert \bar{b} \rvert$	795 (5.98%)
≈ 8	448 (3.37%)
11	216 (1.62%)
Vector-like	177 (1.33%)
$9.23 \approx \sqrt{53+34} = \sqrt{\lvert \bar{a} \rvert^2 + \lvert \bar{b} \rvert^2}$	119 (0.89%)
$7.28 \approx \lvert \bar{a} \rvert$	78 (0.58%)
Other positive number	1,430 (10.76%)
Variable-like	19 (0.14%)
Other	47 (0.35%)
Empty	275 (2.07%)

Table 5. Answer categories for task 2.5.

Task 2.5 categories	Number of students
Correct (2 points)	9,213 (69.35%)
"21i+10j"	2,781 (20.93%)
Variable-like	276 (2.07%)
Other vector-like	167 (1.25%)
31	139 (1.04%)
11	130 (0.97%)
Other positive number	264 (1.98%)
Negative number	130 (0.97%)
Other	53 (0.39%)
Empty	131 (0.98%)

Table 6. Answer categories for task 2.6.

Category for 2.6	Number of students
Correct (105)	4,895 (36.84%)
Sign of dot product (75)	674 (5.07%)
Radians (2 or 1.8 or 1.83)	597 (4.49%)
1	519 (3.9%)
26	237 (1.78%)
90	230 (1.73%)
100	199 (1.49%)
Accuracy (for example 105.02)	175 (1.31%)
Other number	4,698 (35.36%)
"Formula"	48 (0.36%)
"Can not be defined"	15 (0.11%)
Other	102 (0.76%)
Empty	895 (6.73%)

Student profiles for tasks 2.3 to 2.6

Based on the answer categories, we also categorized the students by what kind of answers they gave in the last four tasks (2.3 to 2.6). Each category was assigned a letter, and the categories were then ordered according to their prevalence. Table 7 shows the 20 most common student profiles. For all tasks, O means a correct answer. For task 2.3, the other categories are H for "9i+25j" answers, V for variable-like, and M for other answers. For task 2.4, L indicates having a positive number and I indicates $|\bar{a}| + |\bar{b}|$. For task 2.5, V indicates "21i+10j". For task 2.6, P indicates dot product sign, R indicates radians, Q indicates the angle being 26 degrees, Y indicates the angle being 1 degree, E indicates empty, T indicates inaccurate results, and L indicates other numbers.

Table 7. Top 20 student profiles for tasks 2.3 to 2.6.

Profile	# students	Error type
OOOO	3,799	None – all correct
OOOL	1,396	2.6: Wrong but a number.
OOOP	445	2.6: Sign of dot product is wrong.
OOOR	382	2.6: Angle in radians.
HOVL	313	2.3: and 2.5: Dot products are 9i+24j and 21i+10j. 2:6: Wrong but a number.
OOVL	297	2.5: Dot product is 21i+10j, 2.6: Wrong but a number.
OOOY	230	2.6: Angle is 1.
OOVO	184	2.5: Dot product is 21i+10j.
MOOO	159	2.3: Other mistakes in 2.3.
HLVL	148	2.3 and 2.5: Dot products are 9i+24j and 21i+10j. 2.4: Wrong positive number. 2.6: Wrong but a number.
OIOL	144	2.6: Wrong but a number.
OLOL	143	2.4 and 2.6: Wrong but numbers.
OOOQ	137	2.6: Angle is 26.
OIOO	126	2.4: Computed sum of absolute values.
OLOO	122	2.4: Wrong but a number.
OOOE	121	2.6: No angle computed.
VOOL	120	2.4: Variables. 2.6: Wrong but a number.
HOOL	119	2.3: Dot product is 9i+25j. 2.6: Wrong but a number.
MOOL	106	2.4: Other error. 2.6: Wrong but a number.
OOOT	102	2.6: Incorrect accuracy.

DISCUSSION AND CONCLUSIONS

We focused on questions 1 and 2 because they measure certain basic skills in mathematics: how to solve equations and inequalities, and how to perform basic operations with vectors. The overall performance was good, but the understanding of vectors was weaker than that of equations and inequalities. The performance in task 2.1 (addition of two vectors) was quite similar for all student groups in question 1 (mean ranks varying between 6,256.83 and 6,618.60), though. Task 1.3 was relatively difficult. Although over half of the students were able to solve it correctly, if we look at the students who did not obtain a scalar in task 2.3

or the students who did not obtain a scalar in task 2.5, less than half of the students could solve task 1.3. Generally, students who did not obtain a scalar in tasks 2.3 or 2.5 performed weaker than other students. This suggests that not knowing the notation $|\overline{\boldsymbol{b}}|^2$ or that dot product gives a scalar, do not seem to be isolated problems. However, 46% of the students not obtaining a scalar in task 2.3 had the correct answer in task 2.4. Part of the problem can be the notation and not necessarily mathematical understanding.

When developing the test of understanding vectors (Barniol & Zavala, 2014), the researchers noticed an error with the vector subtraction in the open-ended case: students subtracted the vectors using components and incorrectly added the second component instead of subtracting it or dropped a component. This resembles some of the errors in tasks 2.1 and 2.2.

Dot product was one of the most difficult topics in the TUV test. Instead of obtaining a scalar, many students answered with a vector with the summands of the dot product as its components (Barniol & Zavala, 2014). The phenomenon of arriving at a vector has also been reported elsewhere in the literature (Kusindrastuti et al., 2019). We observed the same phenomenon in tasks 2.3 and 2.5. However, while only 42% of the students in TUV reached the correct answer, we had 70% in task 2.3 and 69% in task 2.5. In these tasks, 15% had a vector as the answer in task 2.3, and 21% in task 2.5. Dot product was also reported difficult for mathematics teacher candidates in (Saraçoğlu & Özge, 2018).

Task 2.6 is about determining the angle between vectors. However, compared with the direction of vector tasks in TUV (Barniol & Zavala, 2014), the type of this task is different. In TUV, the task is to compute the angle between a vector and the real axis, which means it is an application of basic trigonometry. In task 2.6, one had to use the dot product to determine the angle, probably making task 2.6 more similar with questions using formula $AB\cos\beta$ in TUV. In TUV, 54% of the students correctly computed the direction of the vector (in an easier setting than in task 2.6.) and 78% correctly used the equation $AB\cos\beta$ for calculating the dot product. In task 2.6, 37% of the students calculated the angle correctly. This is a smaller proportion of students than in TUV, but the setting was also more difficult.

Some students did not understand the distinction between vector and scalars (Appova & Berezovski, 2013; Tairab et al., 2020), which can be seen in tasks in 2.1 and 2.2.

In general, our results are quite consistent with the results in earlier studies. Since we used a large data set, this improves the reliability of our results. We found similar issues as had been previously identified, but some of the errors were rarer in our data.

ACKNOWLEDGEMENTS

The authors would like to thank the referees and the university proofreading services for helpful comments and criticism, which considerably improved the article.

REFERENCES

Appova, A., & Berezovski, T. (2013). Commonly identified students' misconceptions about vectors and vector operations. In: S. Brown, G. Karakok, K. H. Roh, and M. Oehrtman (Eds.), *Proceedings of the 16th annual conference on research in undergraduate mathematics education, volume 2* (pp. 8–17). http://sigmaa.maa.org/rume/RUME16Volume2.pdf

Barniol, P., & Zavala, G. (2012). Students' difficulties with unit vectors and scalar multiplication of a vector. *AIP Conference Proceedings. 1413*(1). https://doi.org/10.1063/1.3680007

Barniol, P., & Zavala, G. (2014). Test of understanding of vectors: A reliable multiple-choice vector concept test. *Physical Review Special Topics-Physics Education Research, 10*(1). https://doi.org/10.1103/PhysRevSTPER.10.010121

Demetriadou, E., & Tzanakis, C. (2010). Secondary school students' difficulties with vector concepts and the use of geometrical & physical situations. In: History and epistemology in Mathematics Education. In: Barbin, E., Kronfellner, M. & Tzanakis, C. (Eds.): *Proceeding of the sixth European Summer University ESU6.* Verlaf Holzhausen GmbH. http://www.clab.edc.uoc.gr/HPM/ProceedESU6-final.pdf

Harel, G. (1990). Using geometric models and vector arithmetic to teach high-school students basic notions in linear algebra. *International Journal of Mathematical Education in Science and Technology, 21*(3), 387–392. https://doi.org/10.1080/0020739900210306

Jewaru, A., Umrotul, K. S., & Pramono, N. (2021). Senior high school students understanding of vector concepts in mathematical and physical representations. *AIP Conference Proceedings*, *2330*(1). https://doi.org/10.1063/5.0043433

Kusindrastuti, B., Reyes, M., Rakkapao, S., & Prasitpong, S. (2019, November). Examination of Thai freshmen's understanding on vectors using a model analysis technique. *Journal of Physics: Conference Series, 1380*(1). https://doi.org/10.1088/1742-6596/1380/1/012014

Latifa, B., Purwaningsih, E., & Sutopo , S. (2021). Identification of students' difficulties in understanding of vector concepts using test of understanding of vector. *Journal of Physics: Conference Series*, *2098*. https://doi.org/10.1088/1742-6596/2098/1/012018

Sandoval, I., & Possani, E. (2016). An analysis of different representations for vectors and planes in ℝ^3. *Educational Studies in Mathematics, 92*(1), 109-127. https://doi.org/10.1007/s10649-015-9675-2

Saraçoğlu, H., & Özge, K. (2018). Evaluation of teacher candidates' knowledge about vectors. *Asia-Pacific Forum on Science Learning & Teaching*, *19*(2). https://www.eduhk.hk/apfslt/download/v19_issue2_files/saracoglu.pdf

Stewart, S., & Thomas, M. (2009). A framework for mathematical thinking: The case of linear algebra. *International Journal of Mathematical Education in Science and Technology*, *40*(7), 951–961. https://doi.org/10.1080/00207390903200984

Susac, A., Planinic, M., Klemencic, D., & Sipus, Z. (2018). Using the Rasch model to analyze the test of understanding of vectors. *Physical review physics education research*, *14*(2). https://doi.org/10.1103/PhysRevPhysEducRes.14.023101

Tairab, H., Al Arabi, K., Rabbani, L., & Hamad, S. (2020). Examining Grade 11 science students' difficulties in learning about vector operations. *Physics Education*, *55*(5). https://doi.org/10.1088/1361-6552/aba107

Ylioppilastutkintolautakunta. (2020, 5 12). Lopulliset hyvän vastauksen piirteet kevät 2020. https://www.ylioppilastutkinto.fi/images/sivuston_tiedotot/Hyv_vast_piirt/FI_2020_K/2020_k_m.pdf