



The challenges of mathematizing in Swedish early childhood education

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ABSTRACT: Even though there today is a solid view on early childhood mathematics education being of importance, there is a quite diverse view on how this education is to be realized. In many countries, both teaching and play are emphasized in the early childhood education curricula why the question of how to conduct play-oriented mathematics education is relevant. In this study, we empirically elaborate on mathematizing as a possible cross between play and mathematics education. Mathematizing, as developed by the Dutch mathematician Hans Freudenthal, implies seeing the world in the light of mathematics and being able to identify how mathematics contribute to solving everyday problems. In this study we empirically investigate the theoretical construct of mathematizing in a preschool setting, aiming to clarify when and how mathematizing is made possible, and what seem to be the constraints on it. Based on 15 observations of preschool teachers' attempts to implement play-oriented mathematics education, the presence of a true problem and character of teachers' questions stand out as the keys to making mathematizing, as a merge of play and teaching, possible.

Keywords: *early childhood education, mathematics education, mathematizing, play-responsive teaching*

Introduction

Today, there is in research a solid view that early mathematics is important for children's development in general and mathematical learning in particular. At the same time, there are diverse views on *if* and then *how* young children should be taught mathematics (Björklund & Palmér, 2019; Palmér & Björklund, 2016) and whether the word teaching is

to be used at all in connection to early childhood mathematics education (Hunter & Walsh, 2014; Walsh et al., 2017). These diverse views on teaching in early childhood mathematics education become even more diverse when the question whether teaching is to be integrated with or separated from children's play is added. However, a dichotomy between play and teaching is contradictory to the fact that many countries include preschool in their educational systems in which both teaching and play are highlighted as central in the curricula (Björklund & Palmér, 2019; Walsh et al., 2019). However, even if we accept that teaching may take place in early childhood mathematics education and reject a dichotomy between teaching and play, the question of *how* to conduct play-oriented mathematics education remains relevant. For example, in Sweden (National Agency for Education, 2018) and Finland (Finnish National Agency for Education, 2022) the national curricula state that teaching as well as play are to be part of the early education for children 0–5 years old. What is left to the teachers to figure out, however, is *how* to teach in a simultaneous play- and goal-oriented practice. Previous studies have shown that merging play and teaching is a difficult endeavour, with activities often turning out as either play or teaching (Björklund & Palmér, 2019; Palmér & Björklund, 2023). It has shown to be difficult to respect the child's interests and initiatives while at the same time striving to provide the child meaningful experiences that will advance his or her understanding of the world (Björklund & Palmér, 2022a), in accordance with policy documents such as the national curricula mentioned above.

In this article we intend to contribute insights to the challenge of *how* to conduct play-oriented mathematics education. We do this by trying out a consolidation of mathematics education and play-responsive teaching, both theoretically and empirically. Our aim is not to determine whether play and teaching ought to be consolidated, but rather to elaborate on how such a consolidation could be realized. To investigate this, the notion of mathematizing from the famous Dutch mathematician Freudenthal (1968) will be used. Mathematizing means, in brief, the process of making use of mathematical thinking and skills in problem solving where there is an actual need for mathematics. Freudenthal's starting point was that mathematics should be taught so that the knowledge becomes useful for the learner when solving problems in everyday-life situations, which is why all mathematics teaching should be based on the learner's world and experiences. For young children, mathematics then includes experiences of space, shape, and quantities that they encounter in daily activities, such as orienting in and organizing space for playing (furnishing a doll house or creating a map to hide a treasure), making patterns or sharing objects with peers (see Björklund, 2010a; van Oers, 2013). According to the perspective on mathematics education outlined by Freudenthal, mathematics education for young children should then be based on the children's own lived experiences (for example, play), extending these experiences through mathematical inquiry (Gravemeijer & Terwel, 2000). It is however not evident how Freudenthal's writing can be understood in a preschool setting. Thus, in this article we investigate the theoretical construct of

mathematizing in connection to the challenge of *how* to conduct play-oriented mathematics education. Our specific research question is: When and how is mathematizing made possible in early childhood mathematics education, and what seem to be the constraints on it?

Play, learning and mathematics teaching

Play and learning are often described as intertwined based on play being a natural way for children to explore meaning and develop identity in a safe context (Sutton-Smith, 1997). Play and learning have many similarities, such as a directedness towards a goal, rules to play along with and mediation of meaning, which also refer to the notion of teaching (Pramling Samuelsson & Pramling, 2013).

Even though play is always about “something”, its direction is constantly being renegotiated in the metacommunication between the players, who have agency to determine the direction of the play, to negotiate rules and roles, and to take initiatives to develop a narrative for their play. This narrative may include earlier experiences, but also imaginative narratives. Thus, in play, there is a constant movement both toward and away from reality (van Oers & Duijkers, 2013; Pramling et al., 2019). This implies that play may be described as rule-bound at the same time as free (e.g., Vygotsky, 1966).

Fleer (2011) describes imagination as the bridge between play and learning. This means that imagination can be used as a resource in early childhood education, because it makes it possible for children to change meanings of objects and occurrences depending on what the activity needs in order to fulfill the intentions (e.g., pretending pieces of wood are cookies), while the child at the same time is perfectly aware of an object’s dual meaning in “real life” (*as is*) and in his or her imagination (*as if*). If the cookies need to be divided between the play participants (*as if*), the child can use counting words to map one cookie at a time to each participant (*as is*) (see also Palmér & Björklund, 2024). Thus, reality and imagination are not separate but rather dialectically related (Fleer, 2011). Thereby, in play, teachers can make it possible for children to develop the knowledge and abilities that are necessary for the ongoing play activity, which has been shown to strengthen the activity and the child’s agency. However, this requires that the interaction between teachers and children is simultaneously directed at the goal of the play activity and how the child understands the content that is necessary for playing (Palmér & Björklund, 2019, 2023).

Teaching is most often described as an activity being goal-oriented and having predetermined content about which children are to learn (Björklund & Palmér, 2019). But, as the notion of teaching has not been used in preschool to the same extent as in school, it is not self-evident how teaching is to be understood in a preschool context. One

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common illustration of teaching is the didactical triangle, with its three corners symbolizing teacher, child, and content (Hudson & Meyer, 2011). In an educational context, someone (the child) is to be given conditions to learn something (the content), whereby someone (the teacher) is to act in a way that allows these conditions to be realized. However, according to Pramling et al. (2019), teacher and child are not to be seen as counterparts in this triangle but rather as co-learners. To enable such co-learning in the teaching interaction, there is a need to establish a temporarily sufficient intersubjectivity between teacher and child (Siraj-Blatchford & Sylva, 2004; see examples in Pramling et al., 2019), which is based on the teacher being responsive to the child's perspective. Coordinating perspectives, however, does not necessarily mean experiencing "the same", as teaching always implies that the participants (teacher and child) enter and leave the same activity with different experiences and qualitatively different understandings. Teaching on these premises is thereby a challenging activity to conduct (Björklund et al., 2018).

Play-responsive teaching is a pedagogical approach developed both theoretically and empirically in collaboration between researchers and preschool teachers (Pramling et al., 2019). In short, play-responsive teaching implies that the preschool teacher takes part in children's play and responds to the child's initiatives in a developing interaction. Teaching therefore refers to a joint activity, in which both preschool teachers and children are engaged and contribute. Taking this approach, the teacher acts to extend the children's experiences and new perspectives are brought in through alterations, and in this way different contents for learning come to be included as necessary parts of playing, without interrupting the play. One characteristic feature of play-responsive teaching is that the participants continually shift between *as is* and *as if* and relate them to one another. For example, mathematics (*as is*), may become a structuring resource in the children's play through the introduction of a real-world problem managed within the fictional world of play (*as if*).

Responsiveness to the learner's perspective and way of perceiving a content for learning is central in early childhood education as described above, but this is also in the centre of attention regarding teaching specific contents, such as mathematics. A large body of research emphasizes for example the difference between knowing mathematics and teaching mathematics, with mathematical Knowledge for Teaching (MKT) (Ball et al., 2008) being one example of a framework that in six parts describes the special kind of knowledge needed for teaching mathematics: *Common content knowledge* refers to mathematical knowledge used in settings other than teaching, while *specialized content knowledge* refers to qualitatively different mathematical knowledge that is unique to teaching. Studies have shown that preschool teachers with such knowledge and high mathematical self-efficacy are more sensitive to mathematical elements in play (Oppermann et al., 2016). *Knowledge at the mathematical horizon* refers to knowledge of

how mathematics is related to other topics included in the curriculum; in our case, how mathematics may be related to play. *Knowledge of content and students* combines knowledge about one's students with that of mathematics, and similar *knowledge of content and teaching* combines knowledge of teaching with that of mathematics. Finally, *knowledge of content and curriculum* is knowledge about classroom organization and both one's own curriculum as well as those of other grades and subjects. We have argued elsewhere (Björklund & Palmér, 2022b) that teaching mathematics is particularly demanding in early childhood education, as young children's knowledge is often non-verbal and non-graphical. Similar suggestions have been made by, for example, Mosvold et al. (2011).

There are studies in which the teacher (or researcher) identifies mathematical elements in the children's play, which is why children may be considered to learn mathematics while playing (e.g., Björklund, 2007; Cooke, 2022; Franzén, 2015; Gejard & Melander, 2018; Meaney, 2016). These kinds of studies bring forth children's actions and orientations to the material environment, which indeed may be important insights for recognizing learning potential among young children. However, other studies emphasize interaction between teacher and children as contributing to the extension of the children's experiences (e.g., Ekdahl, 2021; Palmér & Björklund, 2023). For example, open-ended questions related to specific math domains have been shown to enhance mathematical learning during play (Trawick-Smith et al., 2016). In line with the latter approach, we believe that it is not sufficient an adult identifying mathematics in an activity if this is not pointed out for the child to discern and explore through language and/or actions. Studies show that many children do not, on their own, attend to numerical features of their surrounding world (e.g., Hannula & Lehtinen, 2005; McMullen et al., 2020). A recent study (Björklund & Palmér, 2022a) found that the attention shown by teacher and child, respectively, often differs; and other studies show that mathematics and play often become parallel processes, with the teacher asking mathematical questions or highlighting mathematical issues that are irrelevant to the play activity (e.g., Björklund et al., 2018).

Theoretical framework

In this article we address the question of how to merge play and teaching in a context that is both play- and goal-oriented. We do this by focusing on when and how mathematizing is made possible in early childhood mathematics education, and what seem to be the constraints on it. To study mathematization, we connect two fields of research as previously described: play-responsive teaching and mathematics education (see Figure 1).

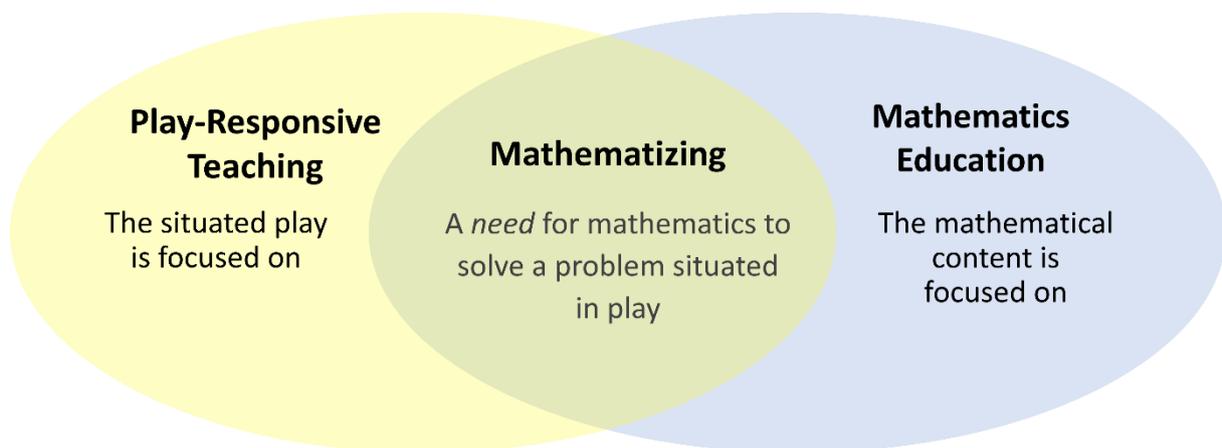


FIGURE 1 Consolidating two fields of research

The cross between play-responsive teaching and mathematics education can be seen as “mathematics made playful” or “mathematizing elements of play” (van Oers, 1996, p. 74). Regardless of the perspective, however, the mathematical content is assumed to be relevant to the child. In line with this, the notion of mathematizing (Freudenthal, 1968) will be used as a theoretical approach, emphasizing mathematics as a part of children’s attempts to make sense of different phenomena (see also Palmér & Björklund, 2024). Freudenthal suggested a change in instructional approach that instead of decomposing and implementing ready-made expert knowledge, students would elaborate, refine, and adjust their current ways of knowing (Gravemeijer, 2004). This instructional approach was expressed as mathematizing meaning, in brief, the process of making use of mathematical thinking and skills in problem solving where there is an actual need of mathematics for the learner to complete a task or solve a problem.

What is mathematics? [...] It is an activity of solving problems, of looking for problems, but it is also an activity of organizing a subject matter. This can be a matter from reality which has to be organized according to mathematical patterns if problems from reality have to be solved. It can also be a mathematical matter, new or old results, of your own or of others, which have to be organized according to new ideas [...]

(Freudenthal, 1971, pp. 413–414).

According to Freudenthal, there is a distinction between horizontal and vertical mathematization where horizontal mathematization means the use of mathematics based on the learner's own life and earlier experiences. In other words, as it appears in the "real world". Vertical mathematization refers on the other hand to a process making it possible to solve a problem by shaping and reshaping symbols which reduces the "noise" that the "real world" may induce. To learn and develop mathematical skills, both are needed, and, in both cases, Freudenthal (1968) emphasized that mathematics should be experientially real for the learners and to be useful in the real world if it is to be taught and learnt successfully. However, "real world" may very well refer to the child's world of fantasy, because the problem to be solved may be an imagined problem occurring within the narrative of the play (van den Heuvel-Panhuizen & Drijvers, 2014; Palmér & Björklund, 2024). In this respect, "real world" may imply different kinds of activities; but it only becomes mathematization if there is a true problem that, from the children's perspective, needs to be solved. For example, if mathematics content is to be taught in the context of play, the mathematics must be useful for the children in the play, as a contribution that helps extend the narrative or solve a problem of a mathematical nature that makes sense in their play. Mathematics education for young children should then be based on the children's experiences (e.g., play) and involve extensions of these experiences through mathematical exploration (Gravemeijer & Terwel, 2000). However, as the notion of mathematizing as developed by Freudenthal was not developed for early childhood education, this theoretical construct needs to be investigated empirically – when and how is mathematizing made possible, and what seem to be the constraints on it?

Method

The empirical material in the article originates from a study conducted in collaboration between preschool teachers from different preschools and researchers from three universities in Sweden participating in a joint research project 2015–2017 (see also Pramling et al., 2019). The collaboration was initiated by preschool principals who asked the researchers for support in developing their educational practices through assigned preschool teachers at each preschool unit. These teachers had special responsibility to develop and spread knowledge in their units about play and learning integrated in teaching (Wallerstedt, 2023). The research relies on authentic video documentation generated by the preschool teachers, where children and teachers engage in play in different ways and with different content. The documentation used in this article involves five preschool teachers enacting play-responsive teaching with the purpose of including mathematics as learning content. The children participating in these documented activities represent the total age span to be found in Swedish preschools; from 1- to 5-year olds. Together, the preschool teachers provided 15 video documentations from

authentic preschool teaching activities. The length of the documentations varies from 2 minutes to approximately 18 minutes. The total video data that has been analysed is 2 hours 27 minutes.

The preschool children's legal guardians gave written consent for the children to participate in video-recorded activities within the regular preschool practice. The children's consent was also carefully considered, with participation in the video-documented play always being voluntary. The participants' names are fictive in all publications (Swedish Research Council, 2017).

To answer our research question, we conducted a qualitative analysis with focus on the interaction between the children, the teacher, and the mathematical content appearing in play activities. In the project, the documentations were first analysed at joint meetings between the researchers and preschool teachers. The guiding questions in these joint analyses were how the teaching was framed in the play, and whether it became possible for the children to learn or develop their understanding of some mathematical concept, depending on how the teachers acted to support their learning. After these meetings the researchers transcribed the observations with attention to the patterns of interaction taking place that included speech, gestures, and use of different materials (see Björklund, 2010b). These transcripts were then used as a basis for analysing the documentations in more detail, focusing on when and how mathematizing was made possible. This analysis was conducted in several steps to identify and interpret patterns of qualitatively similar and different meanings in the data (Braun & Clarke, 2006). First, an overview of the types of activities documented in the data material was created. Then, a closer distinction was drawn regarding sequences in which mathematizing as an analytical lens could be traced, either in the teacher's attempts to initiate a mathematical exploration or in the children's initiatives to explore some mathematical content. In particular, *meaningful mathematics* was one key notion in the analysis, to identify if and how mathematics was made meaningful *from the children's perspective* representing their "real world" (see Freudenthal, 1968). Another key notion was *problem solving*, whether there was a problem in which the children found it necessary to use or develop mathematical knowledge to act out the play activity as they intended, thus identifying the *responsiveness* to the play. Depending on the outcome, as it was expressed in the interaction between the child-teacher dialogues and actions, certain challenges appeared central in explaining when and how mathematizing was made possible. All 15 video documentations were analysed, three of them are used as representative examples in this article to illustrate the result of the analysis.

Results

As mentioned, the documentation used in this article involves five preschool teachers aiming to enact play-responsive teaching with the purpose of including mathematics as learning content. Thus, the teachers made efforts to find activities in which mathematics could be essential and relevant to the play activity or narrative to be acted out. First, three examples will be shown to visualize the possibilities and constraints when including mathematics as part of play. Then, based on these examples, we will elaborate on when and how mathematizing is made possible and what seem to be the constraints on it.

Example 1: “Let’s play breakfast”

“Let’s play breakfast” is a recurring play activity at one preschool that started one morning when the children were having breakfast and spontaneously began engaging in sorting and counting cereal pieces. Later, the teacher picks up on this spontaneous mathematical exploration by bringing the same container of cereal to the table where the children are sitting (not at breakfast time). She frames the activity as “Let’s play breakfast”, and refers to their earlier sorting, counting, and eating of cereal.



FIGURE 2 A teacher introducing a breakfast play activity with five preschool children

This is indeed an interesting way of trying to include mathematics as a relevant feature of children’s play. The relevance is based on playing something that happened before, when the children spontaneously engaged in mathematics as something that appeared to be meaningful (if not useful) to them when eating breakfast. The *play* in playing breakfast is the acting out “as if it were another time of the day”, and in the excerpt below it can also be noted that one child makes the participants’ roles clear before starting the play activity (metacommunication):

Maria (teacher): *Now we're going to play breakfast. Count like we always do; check that we all have the same amount.*

Children: *Maria, you're the teacher and we're kids.*

Maria passes out a handful of cereal pieces to each child, giving them different numbers of cereal pieces – 7, 8, 11.

Maria: *What's the easiest way to count?*

All the children count one by one.

Maria: *Now you counted a little quickly; let's count together. Who has the most and who has the least? How many do you need? You have eight now. If you get two more, how many do you have then?*

The child asks for one more, counts from the beginning again, too quickly arriving at a count of 12.

Maria: *Count one more time.*

A child asks for six more (already having seven) and counts all of them.

Maria: *You ended up with two more than you should have.*

Child: *We're still kids.*

In this excerpt, the teacher introduces specific mathematical tasks concerning the cereal piled on the table. The mathematical content and the tasks introduced by the teacher become counting activities of their own. The breakfast play activity is not visible, and there is no need for the children to know the number of cereal pieces connected to the (invisible) play. They seem happy to engage in this activity and do what the teacher says, but the breakfast play activity is now in the background, the connection lost, even though one child keeps reminding the teacher of their roles in the activity, indicating that the direction of play is being renegotiated through metacommunication.

Example 2: "Making tunnels"

Another example of the teacher taking part in the children's play activity can be seen as a different kind of participation that is more adherent to the children's initiatives and ongoing play. Three boys and three girls are making a tunnel out of cushions at the end of a slide. In this interaction the teacher also poses questions that direct the children's attention to mathematical content. The difference, however, compared to the previous example, is that the questions are posed in order to solve a problem that appears when the teacher takes part in the play activity. This opens up for extending the experiences of how the problem, relating to size and space, can be solved:

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Martha (teacher): *Can you fit under it? Is it high enough? Can I fit under it?*

Children: *Nahh.*

Martha: *How many cushions would it take for me to fit under it then?*

Alex: *Five.*

The children go down the slide into a tunnel made of cushions on the side and a rug on top. One of the children builds it higher using more cushions.

Martha: *Do you think I'll fit under it now? Can I try too?*

The children are worried that the teacher will destroy the tunnel because she is "this big". William shows with his hands held far apart.

Martha: *But what if I crawl in from the other end? She crawls through, did you see? I was able to!*

The children's actions and reactions show that the problem – involving size, space, and how to build using the cushions to also allow the teacher to be a participant – is seen in the worried faces of the children and their reflective comments and attempts to rebuild and revise the play.

Example 3: "Hide the dragon"

The third example involves four children aged two and three playing "Hide the dragon" with their preschool teacher. The teacher instructs the children that one of them is to hide a toy dragon somewhere in the room, and the rest of the children are to look for the dragon with the help of the clue words "up high" or "down low".



FIGURE 3 Mathematical reasoning essential for participating in the play activity

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It is Noah's (child) turn to hide the dragon, and using his hands he shows that he wants to hide it down low. The teacher Malin helps him find a hiding place behind some large cushions under a table (to the right in Figure 3). After he has opened the door to let the other children in, he himself walks directly to the hiding place and points at the dragon. Julia (child) follows him and takes the dragon, but then Noah gets angry. Melvin (child) does not take notice of them, instead looking for the dragon in its first hiding place on a shelf. While the teacher Malin explains to Noah that it is now Julia's turn to hide the dragon, Julia walks out and closes the door after her. The teacher follows her, saying that she should stay inside because it is now her turn to hide the dragon. Julia then takes the dragon and, before the other children have had time to leave the room, puts it on the same shelf as the first hiding place. Noah and Melvin leave the room by themselves, and the teacher walks Andy (child) out of the room.

In a way, this play activity in itself depends on the mathematical notions given as clues by the child who has the hiding role. But this is not what becomes essential in the activity. Instead, the teacher finds it necessary to instruct the children in how to play – the structure of the activity. It is a new activity to the children, and to be able to participate proficiently they need to be able to discern the elements of structure in it. This is a different kind of mathematical content, involving identifying parts of a whole series of acts and roles – who is the hider, who is to look for the dragon, where to hide it (and not to use the same hiding place as before), to hide only when the rest of the group is outside the door, and so on.

Identified constraints on mathematizing

Our observations of teachers making efforts to include mathematics as a content for learning in play-responsive teaching revealed that many observations turned into parallel activities: both a play activity and a mathematics activity, but with the connection between them either not visible to the children or not present (see Example 1). In these situations, mathematizing is not realized because the mathematics is not made useful *for the play to advance or to solve any meaningful problem*. Only a few observations showed indications of mathematics as an included and essential part of the children's play activity and the mathematics thereby becoming relevant to the activity (Examples 2 and 3), thus *mathematics made useful*. When looking more closely at the actions and directions in these observations, we conclude that the following is key: There is rarely a "true problem" of relevance to the play activity to be solved using mathematics, and the mathematical content presented in teachers' questions is often irrelevant to the children's intentions and goals in the activity. But when these keys are present in the teacher-child interaction, mathematizing is realized and play-responsive mathematics teaching is enacted.

True problem

Freudenthal's approach to mathematics education includes that students are to elaborate, refine, and adjust their current ways of knowing based on for them true problems to be solved. The importance of true problem is the first key feature we found in this study. However, we also found that there were rarely any true problems in the play activities that were relevant for the children to solve in a mathematical way. This was observed particularly in play activities with strong guidance from the teacher, which led to the play activity and the mathematical task most often became parallel activities. Many activities can be conducted with no mathematical reflection or reasoning, even though an outside observer might consider some actions to be mathematical. For example, playing with modelling compound (referred to here as "dough") and making rolls, buns, or snakes can easily be done without mathematical references and involve joyful and explorative experiences for the child. But if there is no problem to be solved by mathematical means, it does not make sense to, for example, count pieces of dough or discuss what geometrical shape the "orange" has. To the child, the piece of dough resembles an orange as he or she knows what different fruits look like, and there is no need to know the name of the shape at that particular moment. This becomes evident in our observations of the dialogues between teacher and children, when, for example, a teacher with the best intentions of extending the children's knowledge of shapes asks what shape a piece of dough is and the child answers "dough"; the teacher continues, asking "Dough looking like a...?", and the child answers "ball", followed by the teacher asking "Ball, what does the ball look like?". The child then stops offering any more suggestions. Thus, for the child there is no true problem with an actual need of mathematics to be solved, that is, no need for mathematizing.

On the one hand, if we aim for mathematical learning, the content needs to be brought into the foreground for the children, by metacommunicating what the play is all about, thereby making it possible to expand the meaning and use of tools, preferably by mathematizing. On the other hand, it is quite often the case that attention to shape or quantity has very little relevance in an activity; it is only if you are to build something or, for example, distribute fruit to a number of people, that you might need to know the features of different shapes or how many pieces there are when you have cut the fruit. If we return to the "Hide the dragon" activity (Example 3), the order of actions becomes a true problem to solve (i.e., understand the meaning and implications of) in order to participate. There are various elements that form the structure of the activity that the child must discern and relate to other necessary elements. The mathematical content "structure" (see Mulligan & Mitchelmore, 2013a, 2013b), in one way, is more subtle than notions like high and low; but when looking at the interaction and how the play activity is conducted, the mathematics becomes an essential part and even a necessary aspect to be aware of in order to participate, that is, the key elements of mathematizing. This example

can be seen as mathematizing structural elements of play; not in the sense that the teacher tries to introduce mathematical concepts or operations into the activity but instead in the sense that the teacher tries to make visible the mathematical elements that are necessary for taking part in it. Also, the teacher's participation in building with cushions (Example 2) presents a true problem: If she is to be part of the children's play activity, they have to solve the problem of the size of the cushion construction, as the participants' sizes have immediate impact on whether or not their building actions will work out as planned. In comparison, the cereal counting has no relevance to playing "breakfast" or the children's and teacher's roles that the play narrative includes. It is important to note though, that "real world" and "true problem" may imply different for different children; but mathematization only occurs if there is a "true problem in the real world" from the perspective of the child.

Teachers' questions

The second key feature found for mathematizing concerns teachers' attempts to direct the children's attention to some particular phenomenon. Teachers often use questions to direct attention to certain phenomena or to guide the conversation to some certain new content or a new way of interpreting meaning, or to challenge children in explaining their way of understanding something. This is in line with Freudenthal's suggested change in instructional approach from decomposing and implementing ready-made expert knowledge, to instead making possible for learners to elaborate, refine, and adjust their current ways of knowing. What we observe, however, is that the mathematical content in the teachers' questions is often irrelevant to the play activities, such as when the children are playing with the dough and making "oranges" and "kiwi fruit", which is why the questions "What shape is this [the orange]?" and (when cutting the dough pieces) "How many pieces are there now?" do not make sense in the context of the children's fruit-making activity and are thereby not contributing to children's mathematizing. Again, the actions from the teachers are not connected to any "true problem in the real world" from the perspective of the child.

An interesting observation in the data is that those play activities that were characterized by a high degree of freedom for the children in setting the rules and boundaries more often led to the teachers posing – for the activity – irrelevant questions. Nevertheless, when a teacher is a true participant in a play activity, for example in building a tunnel made of cushions, with the teacher being much bigger than the children (Example 2), questions like "I'm too big; how many cushions should I use so that I'll fit?" become relevant because they need to make her tunnel large enough for her to be able to play (which also relates to having a "true problem"). When the teachers are true participants, they share real world with the children why they may ask questions where there is an actual need of mathematics for the learner to complete a task or solve a problem.

Discussion

As presented, research has showed that attention to mathematics is not evident among all children (Hannula & Lehtinen, 2005). This is one reason why mathematical experiences and basic knowledge and skills to be used in meaningful situations is expected to be part of Swedish early childhood education (e.g., Swedish Agency for Education, 2018). In this article we address the question how to teach in a practice that is to be both play- and goal-oriented, through the notion of mathematizing. To do this, we started in the writings of Freudenthal who emphasised that mathematics education for young children should be based on the lived experiences (for example, play) of the children. Teaching then implies extending these experiences through mathematical inquiry. Our results show how Freudenthal's writing can be understood in a preschool setting but also challenges the idea *how* to conduct play-oriented mathematics education.

The problem we address is both theoretical and empirical as we set out to investigate in what way it is possible to consolidate teaching and play and also how this can be enacted in early childhood mathematics education. This is of relevance as earlier studies have revealed potential for mathematical learning in children's play activities (e.g., van Oers, 2013; Björklund & Palmér, 2022a) but there are observations of teachers' attempts to teach mathematics in play becoming parallel activities (Björklund et al., 2018). The results presented in this article thereby contribute to the ongoing debate about mathematics and play in early childhood education, highlighting in particular the significance of the interaction between teacher and children and emphasizing the importance of teachers pointing mathematics out to children.

As mentioned, interaction and shared attention between teacher and child is essential for children to benefit from teaching where play activities are regarded to be a suitable form for teaching young children. However, to establish the sufficient intersubjectivity that is necessary for play-responsive teaching, the teacher needs to be responsive to the child's perspective and coordinate between it and the teacher's own perspective (Pramling et al., 2019). In addition, the teacher's mathematical knowledge for teaching (MKT) plays an important role for what mathematical content to direct attention to, what certain meaning of a concept that could be explored in a specific situation, and how to use representations or examples that attract the child's attention to the intended meaning (Mosvold et al., 2011). In other words, to make mathematizing possible for the children. To be able to teach mathematics in play – to make mathematizing possible – teachers must have a special kind of knowledge that is necessary for teaching mathematics (Ball et al., 2008). This is observed in the present study when teachers are able to pick up on children's spontaneous initiative, taking it as a point of departure for further exploration and concept development. For example, in the "hide the dragon" play where the intended

learning objects “high” and “low” were abandoned and a different learning object was brought fore (the structure of the play). Knowledge at the mathematical horizon refers to knowledge of how mathematics is related to other topics included in the curriculum; for example, knowledge of how symbols are used in other settings than mathematical ones, and in our case, how mathematics may be related to play. Thus, to be able to make it possible for children to mathematize, teachers need to know not only how children learn mathematics but also how to teach mathematics within the context of preschool. The difficulty of this is observed in the “let’s play breakfast” play where the mathematics seem difficult to make an integrated part of the play, it becomes either play or a mathematical task. It is by all means important to learn to count or know the names of geometrical shapes, but this needs to be introduced and explored when it is found to be necessary for the child, for example in play. In line with this, we find that teachers’ questions and true problems are key features of making mathematizing possible – and, as we suggest based on our empirical observations, key to consolidating mathematics teaching and play. When this is accomplished, any dichotomy between the two disciplines is possible to erase, or at least overcome.

However, when we say that there is no dichotomy between play and teaching, we do not mean that including mathematics in play – in a way that is relevant to the play – is an easy endeavour. Quite the opposite. In the joint project in which we have analyzed teaching situations, the teaching was conducted with the very best intentions of including mathematics as a meaningful part of play-responsive teaching. Nevertheless, it is in the thorough analysis of when mathematizing occurs, and when it does not, that we can chisel out the necessary conditions and keys for success. According to Fleer (2011), *imagination* is the bridge between play and learning. We can see this in our empirical material as well. Even though mathematics as content can be considered *as is*, the true problem and the questions may preferably be *as if* questions, as this allows the mathematics to become embedded in the play narrative. As our observations show, when this happens, mathematizing is made possible (see “hide the dragon” and “making tunnels”) and when *as is* and *as if* are not connected (see “let’s play breakfast”), mathematizing is not realized by the children.

In conclusion, we do see possibilities for conducting play-responsive mathematics teaching and the notion *mathematizing* may be a way to conceptualize how this is realized. But we also see the constraints that make mathematics seemingly difficult to embed as relevant in children’s play. This is a suggested outset for further studies in collaboration with preschool teachers.

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References

- Ball, D., Thames, M., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389–407. <https://doi.org/10.1177/0022487108324554>
- Björklund, C. (2007). *Hållpunkter för lärande. Småbarns möten med matematik* [Critical conditions of learning – toddlers encountering mathematics] [Doctoral dissertation, Åbo Akademi]. Department of Early Childhood Education, Faculty of Education, Åbo Akademi University, Jakobstad.
- Björklund, C. (2010a). Broadening the horizon – toddlers’ strategies for learning mathematics. *International Journal of Early Years Education*, 18(1), 71–84. <https://doi.org/10.1080/09669761003661246>
- Björklund, C. (2010b). Att fånga komplexiteten i små barns lärande – en metodologisk reflektion. *Nordisk Barnehageforskning*, 3(1), 111–120. <https://doi.org/10.7577/nbf.254>
- Björklund, C., & Palmér, H. (2019). I mötet mellan lekens frihet och undervisningens målorientering i förskolan. *Forskning om undervisning och lärande*, 7(1), 64–85. https://forskul.se/wp-content/uploads/2019/03/ForskUL_vol7_nr1_2019_s64-85.pdf
- Björklund, C., Magnusson, M., & Palmér, H. (2018). Teachers’ involvement in children’s mathematizing – beyond dichotomization between play and teaching. *European Early Childhood Education Research Journal*, 26(4), 469–480. <https://doi.org/10.1080/1350293X.2018.1487162>
- Björklund, C., & Palmér, H. (2022a). Teaching toddlers the meaning of numbers – connecting modes of mathematical representations in book reading. *Educational Studies in Mathematics*, 110, 525–544. <https://doi.org/10.1007/s10649-022-10147-3>
- Björklund, C., & Palmér, H. (2022b). Pre-school teachers’ ways of promoting mathematical learning in picture book reading. In S. Dunekacke, A. Jegodtka, T. Koinzer, K. Eilerts & L. Jenßen (Eds.), *Early childhood teachers’ professional competence in mathematics: International perspectives* (pp. 187–205). Routledge. <https://doi.org/10.4324/9781003172529-12>
- Braun, V., & Clarke, V. (2012). Thematic analysis. In H. Cooper (Ed.), *APA Handbook of research methods in Psychology: Vol. 2. Research designs* (pp. 57–71). American Psychological Association. <http://dx.doi.org/10.1037/13620-004>
- Cooke, A. (2022). Identify mathematics in the actions of very young preverbal children when playing with trains. In J. Hodgen, E. Geraniou, G. Bolondi & F. Ferretti (Eds.), *Proceedings of the twelfth congress of European research society in mathematics education (CERME12)*. ERME / Free University of Bozen-Bolzano.

Björklund & Palmér.

Journal of Early Childhood Education Research 13(2) 2024, 167–186. <https://journal.fi/jecer>

- Ekdahl, A-L. (2021). Different learning possibilities from the same activity – Swedish preschool teachers' enactment of a number relation activity. *Scandinavian Journal of Educational Research*, 65(4), 601–614. <https://doi.org/10.1080/00313831.2020.1739131>
- Finnish National Agency for Education (2022). *National core curriculum for early childhood education and care* (Regulation OPH-700-2022). Finnish National Agency for Education.
- Fleer, M. (2011). 'Conceptual play': Foregrounding imagination and cognition during concept formation in early years education. *Contemporary Issues in Early Childhood*, 12(3), 224–239. <https://doi.org/10.2304/ciec.2011.12.3.224>
- Franzén, K. (2015). Under threes' mathematical learning. *European Early Childhood Education Research Journal*, 23(1), 43–54. <https://doi.org/10.1080/1350293X.2014.970855>
- Freudenthal, H. (1968). Why to teach mathematics as to be useful? *Educational Studies in Mathematics*, 1(1), 3–8.
- Freudenthal, H. (1971). Geometry between the devil and the deep sea. *Educational Studies in Mathematics*, 3(3/4), 413–435.
- Gejard, G., & Melander, H. (2018). Mathematizing in preschool: Children's participation in geometrical discourse. *European Early Childhood Education Research Journal*, 26(4), 495–511. <https://doi.org/10.1080/1350293X.2018.1487143>
- Gravemeijer, K., & Terwel, J. (2000). Hans Freudenthal: A mathematician on didactics and curriculum theory. *Journal of Curriculum Studies*, 32(6), 777–796. <https://doi.org/10.1080/00220270050167170>
- Gravemeijer, K. (2004). Local instruction theories as means of support for teachers in reform mathematics education. *Mathematical Thinking and Learning*, 6(2), 105–128. https://doi.org/10.1207/s15327833mtl0602_3
- Hannula, M., & Lehtinen, E. (2005). Spontaneous focusing on numerosity and mathematical skills of young children. *Learning and Instruction*, 15(3), 237–256. <https://doi.org/10.1016/j.learninstruc.2005.04.005>
- Hudson, B., & Meyer, M. (Eds.) (2011). *Beyond fragmentation: Didactics, learning and teaching in Europe*. Barbara Budrich.
- Hunter, T., & Walsh, G. (2014). From policy to practice? The reality of play in primary school classes in Northern Ireland. *International Journal of Early Years Education*, 22(1), 19–36. <https://doi.org/10.1007/s11217-012-9288-5>
- McMullen, J., Verschaffel, L., & Hannula-Sormunen, M. (2020). Spontaneous mathematical focusing tendencies in mathematical development. *Mathematical Thinking and Learning*, 22(4), 249–257. <https://doi-org.ezproxy.ub.gu.se/10.1080/10986065.2020.1818466>
- Meaney, T. (2016). Locating learning of toddlers in the individual/society and mind/body divides. *Nordic Studies in Mathematics Education*, 21(4), 5–28.
- Mosvold, R., Bjuland, R., Fauskanger, J., & Jakobsen A. (2011). Similar but different – investigating the use of MKT in a Norwegian kindergarten setting. In M. Pytlak, T. Rowland & E. Swoboda (Eds.), *Proceedings of the seventh congress of the European society for research in mathematics education* (pp. 1802–1811). University of Rzeszów.
- Mulligan, J., & Mitchelmore, M. (2013a). Early awareness of mathematical pattern and structure. In L. English & J. Mulligan (Eds.), *Reconceptualizing early mathematics learning* (pp. 29–45). Springer.

Björklund & Palmér.

Journal of Early Childhood Education Research 13(2) 2024, 167–186. <https://journal.fi/jecer>

- Mulligan, J., & Mitchelmore, M. (2013b). Reconceptualizing early mathematics learning. The role of pattern and structure. In L. English, & J. Mulligan (Eds.), *Reconceptualizing early mathematics learning* (pp. 47–66). Springer.
- Oppermann, E., Anders, Y., & Hachfeld, A. (2016). The influence of preschool teachers' content knowledge and mathematical ability beliefs on their sensitivity to mathematics in children's play. *Teaching and teacher education*, 58, 174–184.
<http://dx.doi.org/10.1016/j.tate.2016.05.004>
- Palmér, H., & Björklund, C. (2016). Different perspectives on possible – desirable – plausible mathematics learning in preschool. *Nordic Studies in Mathematics Education* 21(4), 177–191.
- Palmér, H., & Björklund, C. (2019). Toddlers exploring structural elements in play. In U. T. Jankvist, M. van den Heuvel-Panhuizen & M. Veldhuis (Eds.), *Proceedings of the eleventh congress of the European society for research in mathematics education* (pp. 2331–2338). Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.
- Palmér, H., & Björklund, C. (2023). The teaching of numbers in common preschool activities – a delicate balancing act. *Early Childhood Education Journal*, 51, 971–980.
<https://doi.org/10.1007/s10643-022-01354-x>
- Palmér, H., & Björklund, C. (2024). The real world of toddler mathematics. In H. Palmér, C. Björklund, E. Reikerås & J. Elofsson (Eds.), *Teaching mathematics as to be meaningful – foregrounding play and children's perspectives* (pp. 221–232). Springer.
<https://doi.org/10.1007/978-3-031-37663-4>
- Pramling, N., Wallerstedt, C., Lagerlöf, P., Björklund, C., Kultti, A., Palmér, H., Magnusson, M., Thulin, S., Jonsson, A., & Pramling Samuelsson, I. (2019). *Play-responsive teaching in early childhood education*. Springer. <https://link.springer.com/book/10.1007%2F978-3-030-15958-0>
- Pramling Samuelsson, I., & Pramling, N. (2013). Orchestrating and studying children's and teachers' learning: Reflections on developmental research approaches. *Education Inquiry*, 4(3), 519–536. <https://doi.org/10.3402/edui.v4i3.22624>
- Siraj-Blatchford, I., & Sylva, K. (2004). Researching pedagogy in English pre-schools. *British Educational Research Journal*, 30(5), 713–730.
<https://doi.org/10.1080/0141192042000234665>
- Sutton-Smith, B. (1997). *The ambiguity of play*. Harvard University Press.
- Swedish National Agency for Education (2018). *Curriculum for the Preschool lpfö18*. National Agency for Education.
- Swedish Research Council (2017). *Good research practice*. Swedish Research Council.
- Swedish Schools Inspectorate (2017). *Preschool quality and aims fulfilled*. Swedish Schools Inspectorate.
- Wallerstedt, C. (2023). Managing the tension between the known and the unknown in knowledge-building: The example of the play-responsive early childhood education and care (PRECEC) project. In C. Wallerstedt, E. Brooks, E. Eriksen Ødegaard & N. Pramling (Eds.), *Methodology for research with early childhood education and care professionals. Example studies and theoretical elaboration* (pp. 45–56). Springer.
- Walsh, G., McGuinness, C., & Sproule, L. (2019). 'It's teaching ... but not as we know it': Using participatory learning theories to resolve the dilemma of teaching in playbased practice.

Björklund & Palmér.

Journal of Early Childhood Education Research 13(2) 2024, 167–186. <https://journal.fi/jecer>

Early Child Development and Care, 189(7), 1162–1173.
<https://doi.org/10.1080/03004430.2017.1369977>

- van den Heuvel-Panhuizen, M., & Drijvers, P. (2014). Realistic mathematics education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 521–525). Springer.
- van Oers, B. (1996). Are you sure? The promotion of mathematical thinking in the play activities of young children. *European Early Childhood Education Research Journal*, 4(1), 71–89.
<https://doi.org/10.1080/13502939685207851>
- van Oers, B., & Duijkers, D. (2013). Teaching in a play-based curriculum: Theory, practice and evidence of developmental education for young children. *Journal of Curriculum Studies* 45(4), 511–534. <http://dx.doi.org/10.1080/00220272.2011.637182>
- Vygotsky, L. (1966). Play and its role in the mental development of the child. *Voprosy psikhologii*, 12(6), 62–76. (Original published 1933).