Experiental Education of Mathematics: Art and Games for Digital Natives

Kristóf Fenyvesi, Raine Koskimaa & Zsolt Lavicza

As attitude-researches point out, students tend to sustain an aversion to mathematics, while remaining largely ignorant of how deeply embedded it is in the world around them. Most students however are able to recognize patterns and numerous research and empirical evidence indicates that they become easily motivated when mathematical connections are presented in ways which relate to their experiences by triggering their natural curiosities. PISA and TIMSS results and recommendations are that students should find education enjoyable, develop self-belief and stamina to address challenging problems and situations. Experience-centered education of mathematics through arts and playful activities might be an effective way to grasp the complex relationship between mathematics attitudes and joy of learning and support the students in their study achievements. In this article we show that creating visual illusions, paradox structures and 'impossible' figures through playful and artistic procedures, holds an exciting pedagogical opportunity for raising students' interest towards mathematics and natural sciences and technical aspects of visual arts. There are certain digital games as well, which employ visual illusions as a part of their game mechanic. Most of these games were not designed as an educational game, but they may be used for educational purposes, to clarify mathematical concepts behind and related to visual illusions (symmetry, perspective, isometric projection etc.).

Introduction

The technologization, digitalization, networkization, and increasing computational complexity of daily practices are reorganizing our society and culture in prolific ways. The increasing importance of mathematically structured systems, patterns, and models has a fundamental impact on our experience of everyday life, and a particular significance for all digitized societies. The abstractness of mathematics as a science, however, makes it a unique discipline often perceived as external to the contexts of daily life. As numerous studies are pointing out, this supposed externality or detachment has raised inconveniences and negative attitudes toward mathematics and mathematical-scientific ways of thinking as such (Malmivuori 2001). While this widening gap between mathematics and society and its paradoxical nature was recognized decades ago,[1] it has continued to grow as recent TIMSS 2011 and PISA 2012 assessments have proven with shocking evidence. The escalating misinterpretation and misunderstanding of the determining technical, economic, environmental, social and cultural processes explored and expressed with mathematical codes can lead to the rapid weakening of social equity and hamper equal access to the controlling systems, technologies, and implicit knowledge of modern society, which form the base of a democratic society (cf. Steen 2001).

As attitude-researches point out, students tend to sustain an aversion to mathematics (Iben 1991; Ma & Kishor 1997; Ruffell & Allen 1998; Gomez Chacon 2000; Hannula 2002; Uusimaki 2004), while remaining largely ignorant of how deeply mathematics is embedded in the world around them (Hannula 2011; 2012; Roesken, Hannula & Pehkonen 2011). Moreover, mathematics traditionally has been regarded as a male domain, which led to gender bias in mathematics performance in education and is known to be a significant part of the issues connected to math anxiety (Curtain-Phillips 1999; Ashcraft 2002).

Most people however are able to recognize patterns and deal fluently with the abstractions of language, music, and visual arts. Numerous research and empirical evidence indicates that people become easily motivated and even fascinated when mathematical connections are presented in ways which relate to their experiences by triggering their natural curiosities and aesthetic sensibilities. There is already significant research made by mathematicians, historians, educators and artists in the exploration of mathematical connections between the arts, sciences, music, architecture and other domains of culture.[2] Expanding interdisciplinary fields of research like visual mathematics, ethno-mathematics, symmetry studies and studies of experiential and inquiry-based learning of mathematics have accumulated an enormous body of results over the recent decades (Fenyvesi 2012; Artigue & Blomhøj 2013). Geometric and mathematical art, from Paleolithic ornaments to contemporary digital art and design, have produced substantial evidence of how deeply mathematical knowledge and systems thinking is embedded in visual culture (Jablan & Radovic 2011). However, these mathematical connections unfortunately rarely enter the school curricula. If they do, they rather appear on the periphery, as an interesting curiosity but not as a central topic or as a part of the core content of mathematics classes.

In this article, first we introduce the results from our recent survey concerning Serbian students' attitude towards mathematics and mathematics education, to provide evidence for the alienation of mathematics from the everyday concerns of adolescents. Then, to resolve this, we propose experience-centered education of mathematics through arts and playful hands-on and digital activities as effective ways for grasping the complex relationship between mathematics attitudes, the joy of learning and social situatedness, and so as to support the students equally in their study achievements.

This approach is in accordance with the consequences of TIMSS 2011 studies and PISA 2012 recommendations. PISA 2012 suggests that students should find education enjoyable and develop self-belief and stamina to address challenging problems and situations. For improving mathematical literacy and abilities, what we believe is important is research on new, experience-centered, art related approaches in mathematics education and the increase of presentations of cultural, interdisciplinary, and artistic embeddedness of mathematical knowledge in mathematics curricula, leading to creative applications of mathematics using hands-on models, digital and mobile tools, virtual environments and the incorporation of real-life problems (or, 'authentic mathematics', Forman & Steen 2000) into mathematics classes.

Serbian Students' Attitudes towards Mathematics and the Visuality & Mathematics Tempus Project

The Visuality & Mathematics Tempus Project

The Visuality & Mathematics — Experiential Education of Mathematics Through the Use *of Visual Art, Science, and Playful Activities (2012–2014)* Tempus Project was initiated by the cooperation of eight European universities and scientific institutions.[3] With an interdisciplinary team of mathematicians, artists, researchers of education, teachers from the secondary and third level education and university students, our goal was to bring about a reawakening of the connections between mathematics and the visual arts in the Serbian mathematics education curricula with interactive, experience-centred, culture- and arts-related content and to develop the conditions of Serbian mathematics education with technological equipment. We recommended various art-connected educational materials, tools, activities, and methods as well as tasks of a playful and creative nature for use in mathematics classrooms.

In the project, we were not only developing genuinely new content and methods for Serbian mathematical education, but also collecting the already existing practices of experience-centred mathematical education, teaching resources, and tools in Serbia. We made our findings and publications available on the project website [4] so as they could be widely disseminated to Serbian mathematics teachers and introduced in teacher training. Also for this purpose, we organized *European Summer Schools for Visual Mathematics and Education* in Eger, Hungary in 2013 [5] and in Belgrade, Serbia in 2014.[6]

During 2013–2014, our Tempus project team conducted 2 rounds of attitude surveys for Serbian students, a wide range group, and their attitudes towards mathematics as such and towards the education of mathematics were measured. The first survey was distributed in 2013. Based on the first round of survey results we offered special training for teachers in experiential education of mathematics and asked them to utilize our methods for a full academic year in their schools. After that we conducted the second survey to assess the effects of the new teaching techniques. A third short survey was conducted in 2013 and 2014, investigating the success of the summer schools organized within the framework of our Tempus project among the Serbian teachers who participated. All the results were used to make specific recommendations for the future development of mathematical curricula in Serbian education of mathematics at all levels.

In the present article we introduce the main findings of the 2013 survey and present only one example from the many experience-centered approach and education materials we have developed in the framework of the project (See Fenyvesi et al. Eds. 2014).

Serbian Mathematics Education from the Perspective of PISA 2012

PISA 2012 reveals that although Serbia — which scored 449 points — steadily improved in mathematics education from 2003 (PISA 2013a, 55), the mathematics performance of 15-year old Serbian students is still statistically significantly below the OECD average. According to PISA's definition of mathematical literacy (PISA 2013a, 37–38), Serbian students fall behind the OECD average in their capacity to formulate, employ, and interpret mathematics in various contexts while they also have difficulty recognizing the role that mathematics plays in the world. From the four overarching areas, which the PISA assessment framework for mathematical literacy makes reference to, it is in *quantity* that the Serbian students score higher than their overall mathematics proficiency scale. Operations in the other three areas of mathematical literacy, i.e., (1) *uncertainty and data*; (2) *change and relationships* (PISA 2013a, 101); and (3) *space and shape* (PISA 2013a, 104), cause even more difficulties for them.

PISA 2012 measured not only the students' performances, but also examined whether and how students' exposure to mathematics content can be associated with their performance. This provides a snapshot of the priorities of Serbian mathematics education policies. The survey has shown that Serbian students' exposure to word problems is under the OECD average (PISA 2013a, 147), as is their exposure to applied mathematics (PISA 2013a, 149), while they have significantly more opportunities to learn formal mathematics content during their schooling (PISA 2013a, 148).

The examination of Serbian students' engagement, drive, and self-beliefs in connection with mathematics learning shows that the index of their mathematics self-efficacy — the extent to which they believe in their own ability to handle mathematical tasks effectively and overcome difficulties — is also relatively low, while their index of openness to problem solving is high, although the latter is not reflected in their mathematics performance (PISA 2013b, 11). Serbian students' intrinsic motivation to learn mathematics is slightly lower than the average as well, but from the survey results it is also obvious that the educational system is not taking full advantage of their positive attitudes and their openness to problem solving. In Serbia, less than 30% of students enjoy mathematics (PISA 2013b, 69).

The picture provided by PISA 2012 on Serbian students' mathematics education and attitudes is further refined by our Tempus Attitude Survey 2013 (TAS 2013). We succeeded in identifying a number of pedagogical practices applied to mathematics education of 11–18-year-old Serbian students, for which we could recommend changes in order to improve and build more efficiently on students' attitudes towards mathematics and thus support them in achieving better results.

Tempus Attitude Survey 2013: Main Findings

Interest in the role of affect in science learning grew in the 1960–70s when education policy-makers faced falling enrolments in science in higher education (Ormerod & Duckworth 1975). In researching the reasons for the decreasing number of science students in 1960–70s mathematics education research on mathematic education, two different foci were apparent: 'mathematics anxiety', and 'attitude toward mathematics'. Studies of attitude are based on two beliefs: attitude toward mathematics is related to achievement, and affective outcomes (such as liking mathematics) are significant *per se* (Rosetta, Brown, Evans & Hannula 2006).

Attitudes strongly influence behavior and involve elements of knowledge and affects, and have a strong impact on education processes as well. Poor attitude towards the sciences are often caused by the way the sciences are presented at various school stages (Skryabina 2000). As research points out, this is not usually the fault of teachers, but arises from bad curriculum design, overloaded curricula, and inappropriate assessment. Although attitudes tend to show consistency and are relatively stable, they are open to change and development, given the right circumstances (Saleh & Swe Khine 2011).

In the Tempus Attitude Survey (TAS) we measured students' attitudes towards mathematics learning in Serbian education. We intended to study what specific aspects of Serbian students' mathematics learning experiences are perceived in a positive light and which cause problems. TAS 2013 mapped the complex relationship between students' mathematics anxiety, mathematics achievement in Serbia, and the Serbian students' perception of the teaching methods applied in mathematics education by their teachers.

The TAS 2013 sample came from 5 elementary and 15 high schools, with children spanning in age from 11 to 18 years. They came from those schools where teachers participating in our Tempus project work, and also visited Tempus Summer Schools. The questionnaire used in the survey was made by the Tempus project team, which was lead by a psychologist

and a researcher of mathematics education. They proposed the initial set of indicators that should be investigated in order to meet the major goals of the survey. The initial list of indicators included wide areas such as:
- everyday approach to mathematics: usage, usefulness;
- general ideas about mathematics;
- general ideas about the education of mathematics;
- students own experience with mathematics;
- students own experience with mathematical education.

The final version of the questionnaire consisted of 5 general demographic questions, 7 specific demographic questions, and 41 survey questions.

2,598 students participated in the TAS 2013. The gender distribution of participants was close to equal (Figure 1), the urban population of students from Serbia was slightly overrepresented, [7] but we were able to reach students mostly from the participating cities (Figure 2).

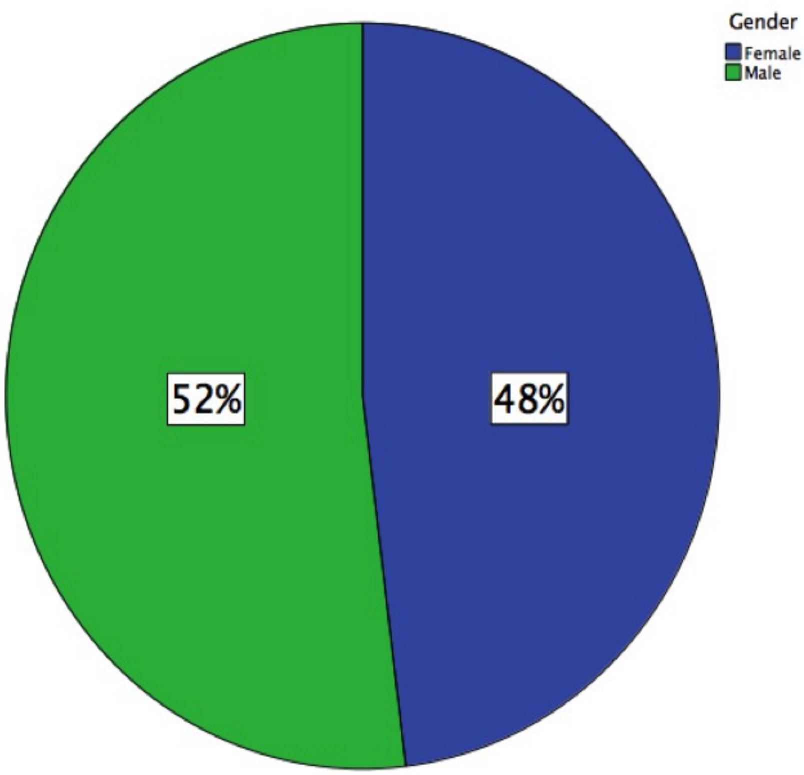


Figure 1: Gender distribution of participants.

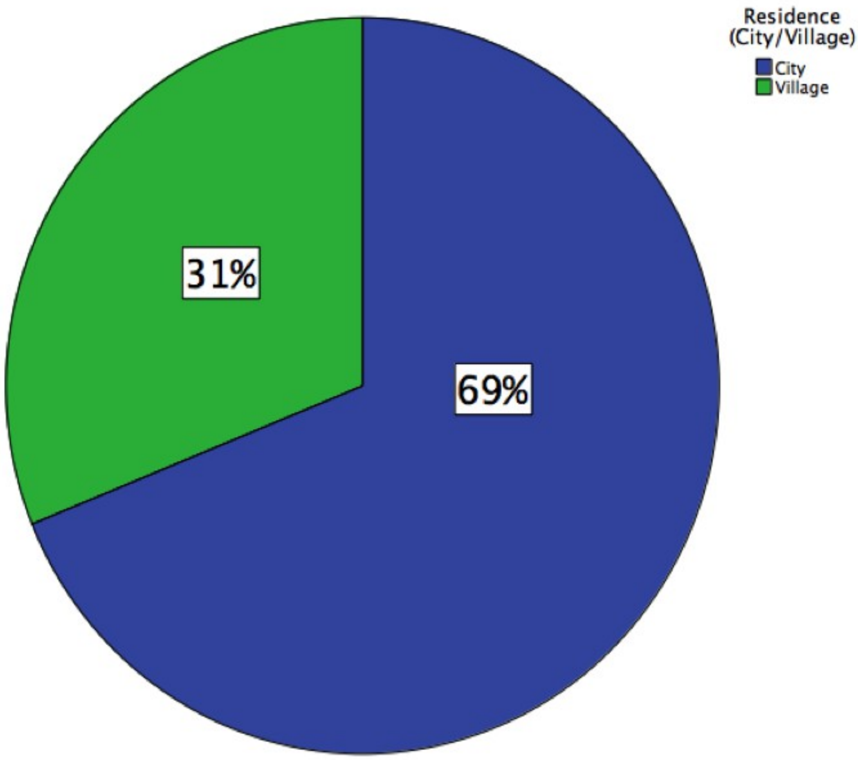


Figure 2: Residence distribution of participants.

While the project was mainly aimed to reach high school students middle school students also participated in the survey. While the majority of students (84%) were attending high school, 16% came from middle schools. The distribution of participants' age can be seen in detail in Figure 3.

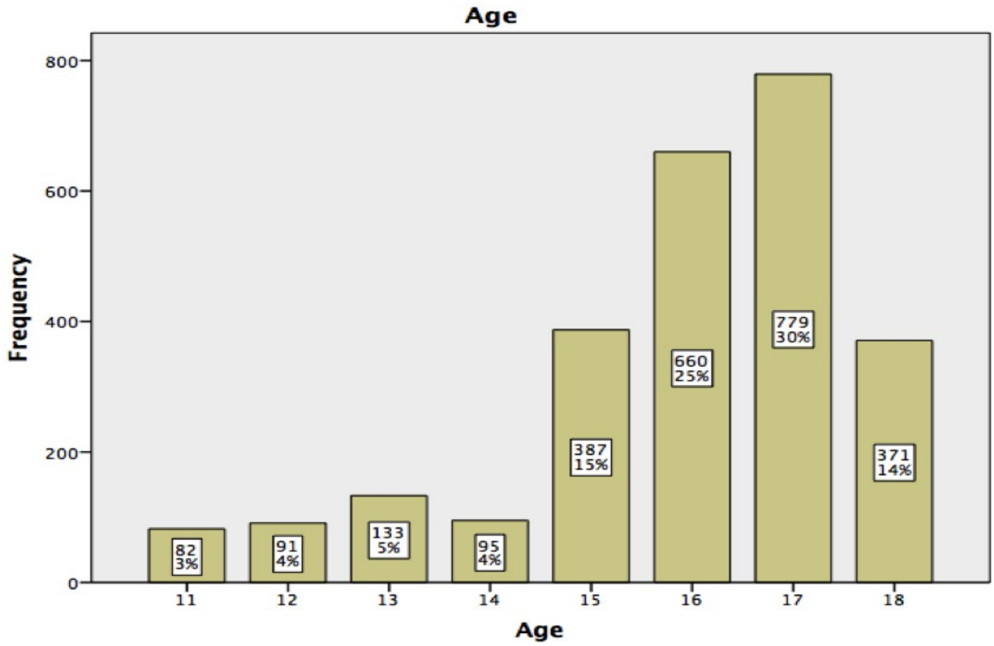
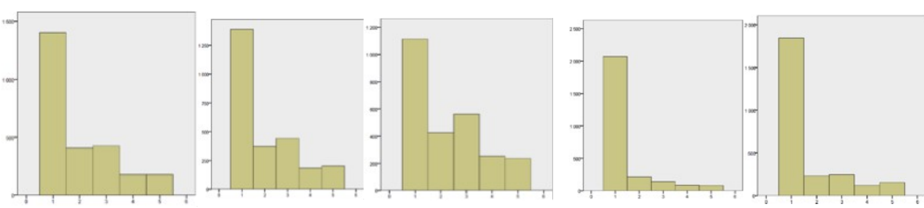


Figure 3: Age distribution of participants.

In the attitude survey we also asked students to inform us about the teaching methods their teachers utilize in their classrooms. According to our results, most Serbian mathematics teachers do not apply methodologies, tools, and equipment for experience-centered mathematics education, which could be effectively implemented to support their students' creative and imaginative abilities in the comprehension of complex and difficult mathematical problems and would make mathematics classes more engaging (Figure 4).



(a) (b) (c) (d) (e)

Figure 4: TAS 2013 results. Students rate how often their mathematics teachers (a) use computers; (b) computer-aided presentations, such as PowerPoint; (c) real physical objects or models for visualization; (d) references to artworks, like paintings or sculpture, etc.; (e) or how often they visited art or science museums to support the understanding of mathematical content. The vertical line shows the number of students; the horizontal line: 1 = never; 2 = a few times;3 = sometimes; 4 = often; 5 = many times.

A third of the students rarely used computers in mathematics classes, and more than half of them never used a computer in their mathematics class. This demonstrates that not only do students not rely on the support of computer applications in math classes, the teachers also use them only very rarely for the illustration of teaching content (e.g., in the form of PowerPoint presentations). The situation is not significantly better in the case of using hands-on tools, physical models, and other visualization equipment: almost half of the students have never had an opportunity to work with these kinds of physical materials in their math classes, or they cannot recall these occasions. The situation is rather unfavorable too in connection to the school presentation of the cultural embeddedness of mathematics. The general mathematics education practice in Serbia almost entirely excludes all accounts of art connections to mathematics. The variety of teaching methods was then complied into three categories expressing teachers' art-related, computer-related, and novel methods of activities. We found that teachers, who use certain experiential approaches or tools frequently, were also more likely to implement other experimental content in their classes. Table 1 shows that there was a high correlation between teachers using different teaching methods. The cross-tabulation of teachers utilizing these methods also supports this result, suggesting that encouraging the implementation of experimental content opens the door to further methods.

	Art Mean	Computer Mean	Methods Mean
Art Mean	1		
Computer Mean	.716**	1	
Methods Mean	.735**	.669**	1
Mathematics is essential for all humans	.096**	.141**	.155**
Mathematics is difficult and hard	-.093**	.141**	.155**
Mathematics improves intelligence	0.019	.072**	.101**
Mathematics is not important for everyday life	-0.015	-.058**	-.079**
Mathematics is the most important course in school	.169**	.169**	.227**
Knowing mathematics opens doors for one's future career	.204**	.203**	.257**
My parents think that it is very important to know mathematics	.082**	.115**	.176**
My parents think that mathematics is important for everyday life	.111**	.128**	.172**
Most students are bored in mathematics classes	-.243**	-.210**	-.224**
Mathematics could be taught in a more engaging way	-.149**	-.076**	-.081**
There are too few mathematics lessons are in school	0.037	.047*	.052**
Most mathematics lessons are very good	.118**	.169**	.222**
We could have learnt more mathematics if it was presented in better way	-.149**	-.171**	-.188**

Table 1: Correlations between teaching methods and attitude scales. (Rating of negatively worded statements were reversed)

It was also interesting to further examine the correlations between students' attitudes to teachers' use of non-traditional methods. It can be seen in Table 1 that students agreed more than not [8] that students believed that mathematics was an important subject for their future careers. The statement suggesting that students were bored in mathematics classes and having negative correlations with the non-traditional methods suggest that such classes and teachers could have a potential to raise students interest and attention in mathematics lessons. Furthermore, the relative high correlations in the statement that teachers offered good quality teaching suggest that students value teachers' efforts at using innovative teaching methods. (For a more detailed analysis, see Fenyvesi et al. 2014.)

We calculated the mean scores of students' responses for attitude items. It can be seen in Table 2 that students mostly agree with most positively stated items and disagree with the negatively stated statements except in the one showing that mathematics was difficult and hard (3.42). It was interesting to observe that students more than not agreed that mathematics improves intelligence (4.15), but at the same time thought that there was a sufficient number of mathematics lessons in schools (1.95). Overall students thought that mathematics was important for everyday life (2.26 reversed) just as their parents (3.84) and that it is also important for all humans (3.72). However, mathematics should be taught more engagingly (3.86). At the same time the relatively high standard deviation (1.446) suggests that some students may not relate mathematics learning to the engaging way of teaching, and there is a diversity of students' opinions about the importance of mathematics in everyday life (1.354).

	N	Means	SD
Mathematics is essential for all humans	2583	3.72	1.142
Mathematics is difficult and hard (-)	2591	3.42	1.235
Mathematics improves intelligence	2585	4.15	1.072
Mathematics is not important for everyday life (-)	2591	2.29	1.354
Mathematics is the most important course in school	2587	3	1.264
Knowing mathematics opens doors for one's future career	2585	3.18	1.195
My parents think that it is very important to know mathematics	2580	3.84	1.164
My parents think that mathematics is important for everyday life	2589	3.66	1.235
Most students are bored in mathematics classes (-)	2587	2.35	1.237
Mathematics could be taught in a more engaging way	2586	3.86	1.239
There are too few mathematics lessons are in school (-)	2582	1.95	1.166
Most mathematics lessons are very good	2580	3.37	1.266
We could have learnt more mathematics if it was presented in better way	2585	3.06	1.446

Table 2: Attitude means and standard deviations.

Interestingly, we did not find much difference between the comparison of boys and girls in the attitude scales, despite the fact that girls were expected by the researchers – based on the wide international literature of the topic – to receive slightly higher grades (mean grade girls=3.5 and boys=3.3) than boys in our study population. It can be seen in Figure 5 that a larger percentage of girls received 4 and 5 grades from the participants.

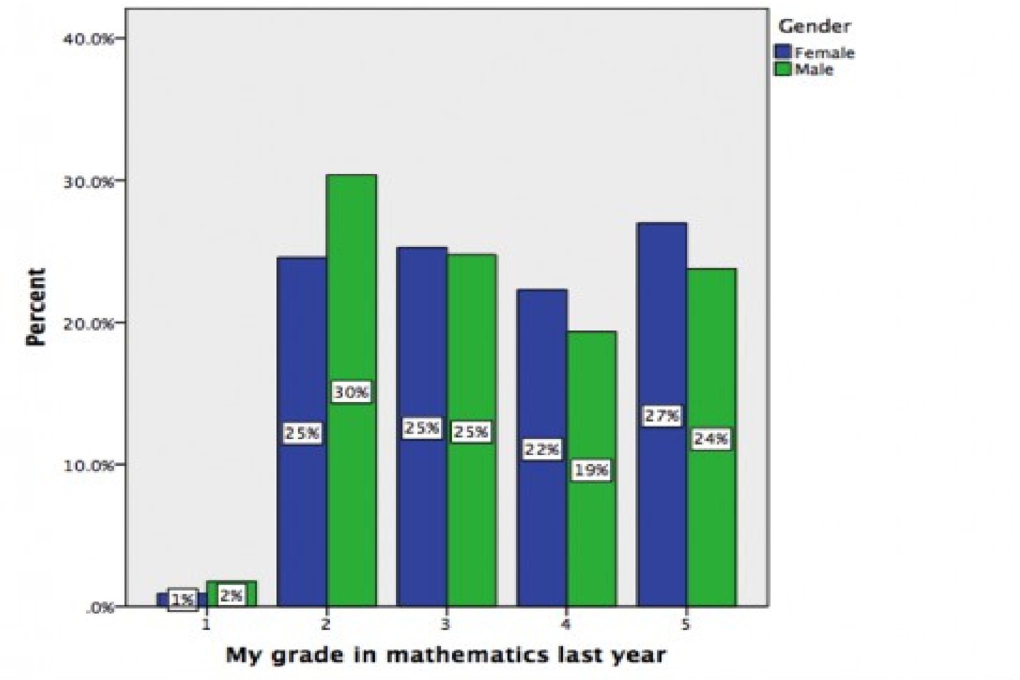


Figure 5: Distribution of grades by gender

To summarize results we can suggest that teachers using more innovative methods in classes could inspire students to learn more and increasingly insightful mathematics. Students acknowledged that mathematics was an important subject in schools and in everyday life and that it has an impact on their future careers. However, to learn it further, they believed that it should be made more interesting, more based on experiences, and be more applicable to them.

Although, from the above, it can be concluded that TAS 2013 main results are in line with the findings of the comprehensive international research in students' attitudes to and beliefs in mathematics, at least four aspects of the results in this article are novel and necessary: (1) our study has addressed a much wider age group of Serbian students than have the PISA and TIMSS surveys; (2) in terms of the wide age group and the number of students who participated in the survey, TAS 2013 was one of the largest studies on students' attitudes toward mathematics ever carried out in Serbia; (3) PISA and TIMSS have not studied students' perception of the teaching methods directly, as did TAS 2013, especially focusing on the presence of experience-centered and “learning through the arts” (LTTA) approaches in Serbian schools; and finally, (4) TAS 2013 has provided for the first time concrete evidence that mathematics education practices in Serbia almost entirely excludes all accounts of art connections to mathematics. This seems to be an overlooked opportunity, which we recommend be changed, taking into account the complex impact that LTTA approaches can have on mathematics education, as is shown by significant international examples (Elster & Ward, 2007).

"Learning Math Through the Arts"[9] : Experiential Workshops

The Experience Workshop Movement in the Visuality & Mathematics Tempus Project

The *Experience Workshop – Movement for the Experience Centered Education of Mathematics* started in 2008 at the *Ars Geometrica Conferences* (2007–2010, Hungary) as a collaborative effort of mathematicians, artists, and teachers of mathematics and the arts. In the open network of the *Experience Workshop Math-Art Movement* almost two hundred scholars, artists, teachers of various subjects, craftsmen and toymakers experiment with various new educational approaches to develop interactive and play-oriented combinations of mathematics and arts. Their aim is to involve the students and their teachers, even their families, into a vibrant dialogue between the mathematical and artistic processes and raise their own personal interests in the field where mathematical and artistic thinking and practice merge.

The *Experience Workshop Movement* (EWM) organize math-art festivals, workshops, exhibitions for children and their parents, trainings and conferences for teachers and professionals interested in an experience-based mathematics education. EWM's events take place in schools, universities, cultural institutions and public places and are widely known and popular all over Hungary and in the neighboring countries, including Serbia.[10]

Recently, more than twenty thousand pupils from primary, secondary, and high schools and universities as well as over two thousand teachers and parents took part in EWM's programs. The *Visuality & Mathematics Tempus Project* has provided an opportunity for the EWM to strengthen its bonds to the Serbian math-art community and to initiate new collaborations with Serbian scholars and teachers of mathematics and arts on all levels of Serbian education.

The *European Summer School for Visual Mathematics and Education*, which was organized two times in the framework of the Tempus project — once in Eger, Hungary in 2013 [11] and once in Belgrade, Serbia in 2014 [12] — offered an opportunity for EWM specialists to meet with a great number of Serbian mathematics teachers and university students and to train them in experience-based mathematics education through arts and playful activities. In the summer schools, we mainly focused on the questions: (1) how to integrate the artistic and cultural connections of mathematics and playful, experience-based approaches into mathematics teaching programs?; (2) how to integrate experience-based mathematics education into art teaching programs?; (3) how to expand the set of tools used for developing students' aesthetic sensibility together with increasing their mathematical, logical, combinatorial and spatial abilities as well as their structured thinking skills?; (4) how to motivate collaborative problem solving, interdisciplinary, and inter-artistic approaches?; (5) how to organize math-art popularization events in Serbian schools to disseminate experience-based education approaches? The Summer School presentations, seminars, workshops and Public Days – organized as mathematics and arts popularization programs for the inhabitants of the hosting city by the summer school participants themselves at public places – led to mutual exchange between the EWM specialists and the summer school participants. Through this, summer schools not only contributed to the development of genuinely new content and methods in Serbian mathematics education, but they also served as platforms for collecting those practices in experience-centered mathematics education, which were already existed in the Serbian math-art-education discourse.[13]

Figure 6: Serbian teachers promoting experience based mathematics education in Eger downtown at the Public Day of European Summer School for Visual Mathematics and Education in 2014.

Math-Art Workshops in the Classroom[14]

EWM's math-art workshops are based on the active and creative manual participation of the students. EWM's programs include such experimental, practical workshops in playful forms relying on mathematical connections in the arts, which exceed the mathematics curricula taught in ordinary schools. As TAS 2013 results have shown, most students require such new approaches in mathematics learning. In EWM's programs, according to their individual choice the pupils can become acquainted with mathematical and artistic procedures through artwork based games and various math-art educational tools developed originally by EWM members. EWM events featuring programs with topics like planar and spatial tessellations; collaborative construction of complex spatial structures (e.g., 3D projections of multidimensional objects, including crystals, fullerenes, quasicrystals, or molecular structures, or even spacebases [Kabai et al. 2012]) with ZomeTool, 4dFrame, and other math-art toolkits. EWM has a wide selection of educational tools and a large international art collection [15] to develop spatial and compositional skills based on the playful recognition of symmetries; EWM also provides many ways to demonstrate non-Euclidean geometries with the help, among others, of Lénárt-spheres; making simple Möbius strips, then more complex one-sided surfaces in different sizes; creating self-similar fractals; analyze and work with the artworks of Escher (Darvas & Fenyvesi 2009; Darvas 2010b), Vasarely (Jablan & Fenyvesi 2011), and so on; movement and dancing, experiments with musical instruments, etc. Symmetry principles are used as a common guide to all these activities based on repetition, algorithms, isotropy, etc. All these approaches, tools, and activities extend the standard teaching programs and develop the creative thinking of the students by burdening their left and right cerebral hemispheres, more or less equally balanced, and by facilitating interaction between the two hemispheres of students' brains (Darvas 2007; 2010a; Leikin et al. 1995; Leikin 2006).

The creative artistic approaches enable students to familiarize and better understand the abstractions and algebraically formulated regularities of mathematical thinking while also contributing to their skills in working with abstract notions and applying systems thinking in problem-solving and decision making. EWM's events mobilize synergies with a multidisciplinary approach. EWM's workshops extend the regular classroom instruction in — at least — two essential ways: in their methods and in their thematics. By providing opportunities for the teacher to experiment with the role of a facilitator, EWM's workshops also let the students solve mathematical problems through playful participation and hands-on activities. Students and teachers, while testing their own creativity, perform such skills and abilities, which have remained latent in traditional classroom processes. E.g. traditional geometry education in Central-Europe is based partly on axiomatic geometry, partly on proving geometrical theorems through algebraic calculations, and partly on construction by ruler and compass. It provides 'dry' knowledge, and is not very attractive for creative pupils. Geometry is basically a left cerebral hemisphere governed knowledge, which demands spatial co-ordination, spatial manipulation, activating the artistic-creative fantasy of the children. Traditional European, Euclid-based geometry teaching – in contrast to e.g., Japanese wasan (*Wasan* 1997) – concentrates on rational elements of geometry, uses algebraic methods, and brings up to a lesser extent the spatial orientation from the students' mind. While the former is based on left hemispheric activity, the latter demands the activation of the right hemisphere of the human brain. EWM's workshops with their hands-on games, connections to art, and creative activities with modeling kits, make students experimenting with the structure of space, shapes, spatial relations, connectivity, proportions, colours and mathematical beauty

Figure 7: Collaborative work with ZomeTool kit at an Experience Workshop event. Photo: G2foto.

Impossible Figures and the Power of Visual Paradoxes: an Example from Experience Workshop's Repertoire

According to Margo Kondratieva, paradoxes, and especially visual paradoxes, are potentially useful for teaching mathematics due to their engaging power and the effect of surprise (Kondratieva 2007; Kondratieva 2009). Kondratieva also sees visual paradoxes as highly useful in classroom as they can be easily implemented as exercises where the pupils can experiment with alternative solutions through drawing or manipulating cut-out shapes. Similarly to EWM's approach, Kondratieva emphasizes the importance of hands-on activities:

> *Manipulations with physical models and figures of geometrical objects allow learners to get a better understanding through reorganization of the perceived information and construction of an appropriate structural skeleton for a corresponding mental model.* (Kondratieva 2009, 4.)

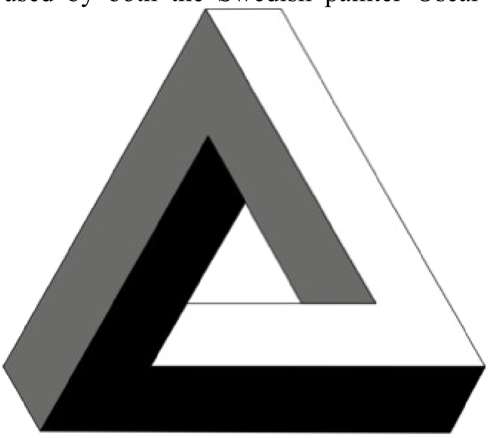
Even though there are clear benefits in this sort of visual experimentation, it is equally evident that the power of visual reasoning is restricted in some aspects. For example, negative and complex numbers cannot be dealt with and are excluded as topics. This limitation may, at least to some extent, be overcome when physical objects are replaced by manipulating virtual objects in digital environments. Another limitation is the reliability of visual images: we cannot necessarily always rely on our own eyes, as various well known visual illusions make this evident. Visual illusions and paradoxes, however, may be turned into means of engagement, and pedagogical tools in themselves.

The key in the visual approach is to foster an easy and fast way to try out several alternative solutions to the given problem:

> *[- -] the point of the exercise is to make a large number of observations, to learn how to make a picture talk to you about its properties, to retrieve the information compressed in a drawing. (*Kondratieva 2009, 5.)

There are several visual artists, teachers, and mathematicians in the EWM's community who work on visual paradoxes and their pedagogical implementation in the experience-based education of mathematics. A special group of visual paradoxes and illusions, namely the impossible figures, are apparently enjoying special interest and are receiving special attention in the EWM's community, with many EWM members developing their pedagogical application. Artists Tamás F. Farkas and István Orosz create impossible figures as a part of their artistic oeuvre, Ildikó Szabó, a mathematics teacher, develops a math-art education program based on Farkas' and Orosz's artworks, and the mathematician László Vörös carries out geometrical research connected to Farkas' and Orosz' art pieces.

Bruno Ernst defined the impossible figures as figures which can be imagined or drawn, but which cannot be made in any concrete form (Ernst 1986). Their effect is based on (at least) two separate layers of illusion. As Ernst summarizes, the first layer is the illusion of spatiality: all we are really looking at is a set of lines printed on a piece of paper (flat), yet we appear to see a solid object. And the second layer is the illusion of continuity: the bars which make up an impossible tri-bar cannot meet in real space (different perspectives united in an isometric drawing), but we still try to assign a meaning (Ernst 1986, 10–15). There are several noted examples of impossible figures from the fine arts, certainly the most famous ones are the Dutch artist M.C. Escher's *Belvedere* (1958), *Ascending and Descending* (1960), and *Waterfall* (1961), but the phenomenon is equally fascinating and challenging to psychologists and mathematicians too. Impossible figures were first described scientifically by psychiatrist Lionel Penrose and his son, the later world famous mathematical physicist, Roger Penrose, in their paper: "Impossible Objects: A Special Type of Visual Illusion," published in the *British Journal of Psychology* in 1958. The paper included illustrations such as the impossible triangle and the impossible steps, both of which were also used by both the Swedish painter Oscar Reutersvärd and M.C. Escher in their works.



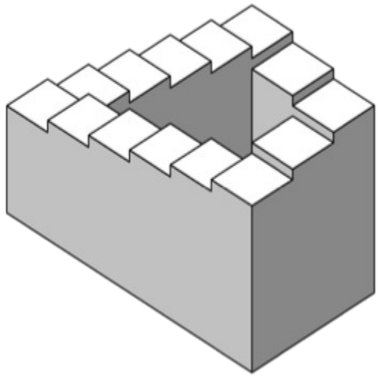


Figure 8: Penrose's impossible trainagle. Figure 9: Penrose's impossible steps.

In the case of impossible figures, a specific correspondence develops between the two- and three-dimensional space. Therefore studying or drawing these figures can play an important role in visual art studies as well as in mathematics education. Studying impossible figures not only helps in thinking creatively but it also improves depth perception. Furthermore, getting acquainted with impossible objects can open the way to understanding higher (more than 3) dimensional spaces and high-dimensional structures within them.

Figure 10: Tamás F. Farkas's compositions with impossible figures.

At Farkas's EWM workshops, students use the artists' templates to recreate his impossible figure designs. The templates are based on the connection between the structural properties of impossible figures and tessellations with special modules, called Necker or Koffka cubes. The Necker cube is an optical illusion of perceptual inversion first published as a rhomboid in 1832 by Swiss crystallographer Louis Albert Necker. Some decades later, the German psychologist Kurt Koffka, one of the founders of Gestalt psychology re-discovered reversible figures like Necker Cube, as a part of his experiments on problem-solving and creativity. As EWM's leading expert of visual mathematics, Slavik Jablan writes in his seminal article "Modularity in Art", Necker or Koffka Cubes are "multi-ambiguous" objects: "they can be interpreted as three rhombuses with joint vertex, as convex or concave trihedron, or as a cube. If we accept its 'natural' 3D interpretation – a cube – then for a viewer there are three possible positions in space: upper, lower left, and lower right, having equal right to be a point of view. So, for the corresponding three directions, a Koffka cube represents a turning point. Having such multiple symmetry, it fully satisfies the conditions to be a suitable basic modular element." Jablan also calls the attention on the connection between the Koffka cube and Thiery-figures (proposed at the end of 19th century) consisting of two Koffka cubes, Reutesvärd's impossible objects, the Penrose tribar, and artworks by Victor Vasarely, among other examples. All of them could be derived as modular structures from a Koffka cube, as "from Koffka cubes we could construct an infinite family of impossible figures. In the process of their growing, in every point, we have a possibility to proceed in three directions, i.e. to choose each from six oriented ways" and exactly this is the underlying principle of Farkas's impossible designs. As Jablan concludes in the same article, "if we introduce in our game Archimedean (or uniform) plane tilings, we could obtain an infinite collection of (possible) and impossible figures, beginning from the elementary ones, and including more sophisticated forms, similar to that occurring in the book *L'aventure des figures impossibles* by B. Ernst, or to the artistic creations by T. Farkas." (Jablan 2005, 263–264.)

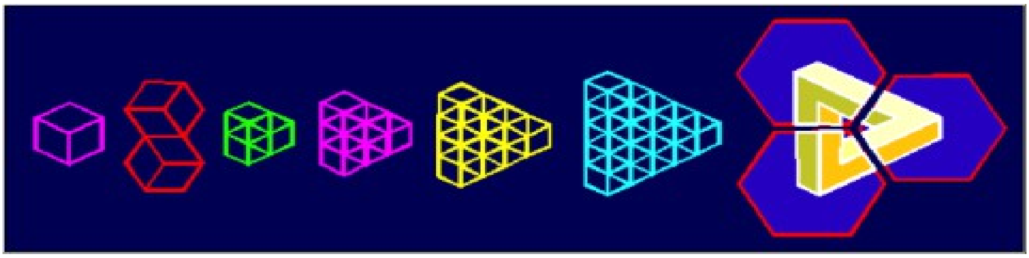


Figure 11: The "evolution" of the Penrose tri-bar from tessellated Koffka cubes.

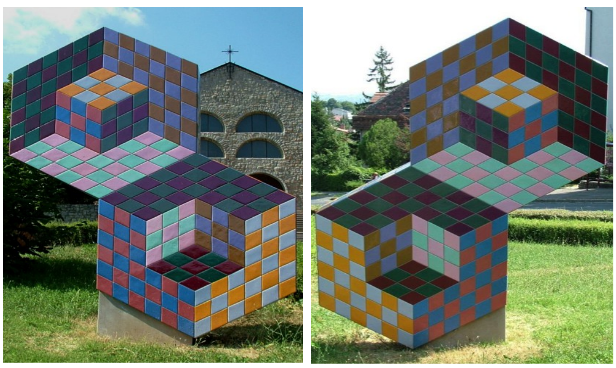


Figure 12: Koffka cubes on Victor Vasarely's JEL sculpture. (Pécs, Hungary, 1977)

F. Farkas's workshop starts with the deep study of his impossible artworks and a free discussion on their gradually discovered geometrical properties. Then each student chooses a figure which they would like to re-create.

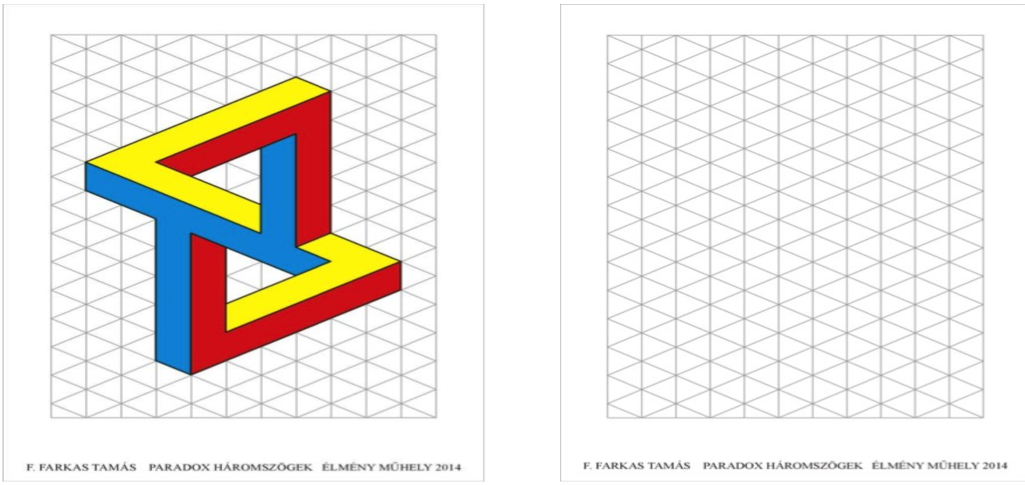


Figure 13: Tamás F. Farkas's templates for creating impossible figures with students. Template A (left), Template B (right). The two Penrose tri-bar of the composition on Template A are diagonally symmetric.

Copies of templates A and B belonging to the given artwork are printed according to the number of participating students. The figure on template A is cut into parts along the black line bordering elements with a pair of scissors. Afterwards, the students' task is to recompose the figure on the raster net B belonging to the given form. The facilitator of the activity might draw the participants' attention to the fact that two elements of the same colour cannot border each other. After completing a figure the participants give a verbal description of the object, defining their specific geometric features and discussing observations obtained during the construction together. The raster net B can also be used by students to draw the figure as well. After becoming familiar with the geometric features of impossible objects, students try to design their own impossible objects on the raster net B, by implementing their geometrical knowledge, developed at the workshop.

Figure 14: Building a "Koffka" pyramid as an introductory exercise in the Impossible Figures workshop with lower primary school pupils in F. Farkas Tamás's workshop in Experience Workshop — International Movement of Experience-Centred Mathematics

Education (www.experienceworkshop.hu) event at ANK School in Pécs. Photos: Csaba József Szabó.

Figure 15: Modeling impossible figures with various math-art tools (MathMaker, JOVO, Jomili cubes) at EWM's pedagogical coordinator Ildikó Szabó's mathematics class.

The experience-centered process of exploratory introduction to geometry problems related to impossible figures can be successfully supported by using Dynamic Geometry Softwares (DGS) such as the free-access GeoGebra (www.geogebra.org) to extend investigations and foster deeper understanding of impossible figures' geometrical properties. GeoGebra is accessible, engaging, encourages students to further explore the geometrical situation, and provides opportunities for making and evaluating conjectures of geometrical results. Students can construct the image of Farkas's impossible figures in GeoGebra and be used to study such questions as e.g. how many different shapes can be seen in the image (different colours, but same shapes not to be regarded as different)? What transformations have to be applied to re-create a figure from single modules? What kind of symmetries can you identify in each figure? etc.

Digital Games Based on Visual Paradoxes

Mathematics educational games are another option to introduce experiential approaches to mathematics teaching. They differ from the exercises described in the previous chapter in that they do not involve such hands-on connection to physical materials, but provide experiential practices through manipulation of virtual objects and environments (the similarity of educational computer games, and hands-on approaches have been emphasized, amongst others, by Squire 2005, 5). Whereas a digital game may lack some in the concreteness of the manipulation, they make the exploration of the situation and its specific characteristics even more easy and engaging, thus helping the pupil to build a strong understanding of the problem in much the same vein James Paul Gee is describing under his notion of "performance before competence" in regards to educational gaming (Gee 2005, 13).

There are many games, both educational as well as entertaining ones, with potential in this field and in what follows we discuss just two of them, *The Bridge* (by Ty Taylor and Mario Castañeda 2013) and *The Monument Valley* (ustwogames, 2014), which are both based on Escherian visual paradoxes. They both use visual paradoxes as game mechanics, and may also be used to facilitate proof construction.

Games like *Monument Valley* and *The Bridge* (see Figure 15 and 16) pose challenges based on visual paradoxes. The player frequently faces situations where proceeding is apparently blocked. There are pathways ending abruptly, staircases leading to solid walls, and targets placed on such positions where no path exists. In order to proceed, the player has first to identify potential paradoxical structures. The player has to observe the game environment and decide which are the most promising elements offering the needed scaffolding. Then, the options provided by the game interface have to be experimented with in order to find a way to manipulate the game world successfully. It is usually considered bad game design if the player has to recede to the strategy of going through all available options more or less systemically in order to eventually stumble onto the right solution, but from an educational perspective, even this kind of mechanic approach bears merit in helping the player to see the different aspects of the visual presentation. When the design is successful, the initial proceeding by surprises-through-mechanic-selections gives way increasingly to proceeding-through-reasoning when the player grasps the logic of the particular visual paradoxes employed in the game.

When the player experiments with the game environment, she builds up her understanding of the problem, or, as Kondratieva formulates it, she is "making a large number of observations, making the 'picture talk'", as she is "search[ing] for the flaw in the initial understanding of the situation" (cf. Kondratieva 2009, 5). Intuition often helps in choosing the most promising directions in the initial phases of problem solving, but it is the very nature of visual paradoxes (as of paradoxes in general) that they are counter-intuitive. The process of going through a number of various alternatives in a systematic way, not precluding any alternatives but experimenting also with attempts that by first sight seem simply impossible, bears two kinds of pedagogical potential. First, it is a way to build up an understanding of how a particular visual paradox is created, but even more importantly, it helps to build up a wholly new understanding of the world surrounding us, forcing us first to reject the naturalistic assumptions and then expanding the pupil's skills to the extent that the solution can be found purely through reasoning. When this point is reached, the step necessary for deductive processing is considerably eased:

> *[- -]visual paradoxes helped students to develop a sense of the purpose of proofs by examining the links between the given information and the conclusion — the core of any deductive process. Their ability to understand and validate logical arguments was enhanced by the search for a flaw in the reasoning leading to a false conclusion.* (Kondratieva 2009, 8.)

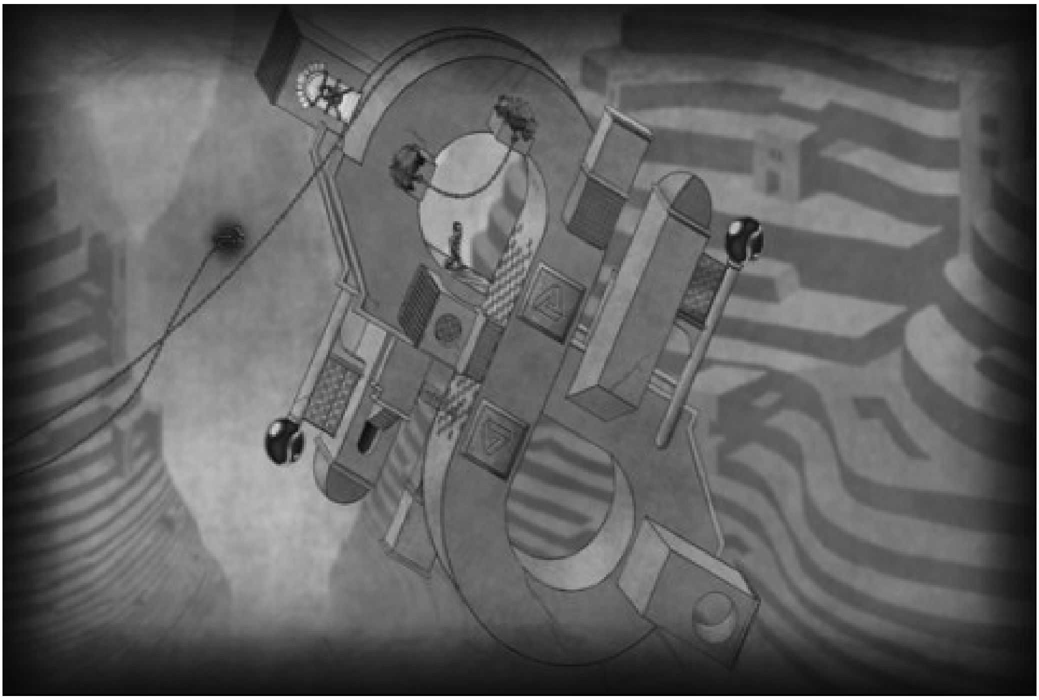


Figure 16: The Bridge is Black and White (image from the game website http://thebrid-geisblackandwhite.com/ - retrieved on 18 December2014)

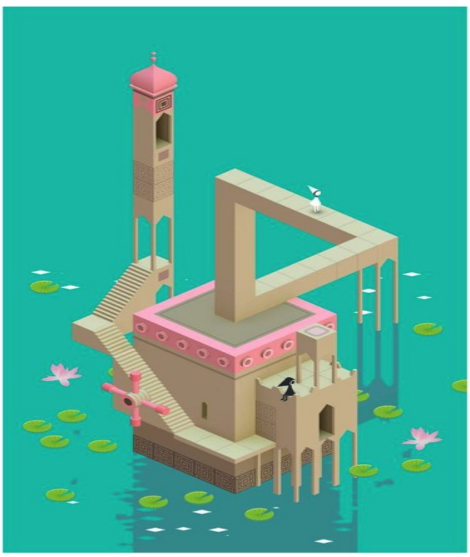


Figure 17: The Penrose tri-bar appears in The Monument Valley (image from the game website: http://www.monumentvalleygame.com/- - retrieved on 18.12.2014).

Thus, the game helps pupils to understand the particular problem, and consequently, to construct a corresponding mental model making it easier to understand how the formal proof is constructed. The game as such, however, does not teach the construction of formal proof itself, but the teacher is there to do this. Together the experimental workshops and digital games provide a rich set of approaches to overcome the attitude challenges in mathematics teaching, and to cater to an even wider variety of different learner types.

Conclusions: Artful Approaches, Playful Attitudes

Our collection of experience based mathematics education materials that support teachers in teaching mathematics through art was just published in English and Serbian (Fenyvesi et al. eds. 2014) and is being distributed among the teachers who participated in our Tempus Summer Schools during the very submission of this article. The attitudes of the Serbian teachers were assessed as well after the first year of the project and there was a clear indication of positive changes taking place through the intervention (Fenyvesi et al. 2014). Regarding the students' learning outcomes, we imagine the results will be in accord with international experiences (Elster & Ward 2007), which have been positive, but it will require further assessments.

The connections between mathematics and the arts, the creative and practical application of these connections and, last but not least, the teaching of mathematics using an interdisciplinary and inter-artistic approach, have a rich modern tradition and an extensive international system of institutions. However, in most cases, the mathematical evidence rarely enters the field of cultural studies, remaining – like most characteristic mathematical features and content of mathematical art – without systematic analysis by scholars of culture and the arts. Regardless of the often recalled traditional collaboration between mathematicians and artists "from the renaissance", and regardless of the alarming notices of the ascending gap between mathematics and society today, there is still no sufficient mutual dialogue between the mathematicians and the scholars of culture and the arts.

As new generations are "growing up digital" and youth cultures are sources of technical cutting-edge applications, educators must recognize students as inventors of culture (Tapscott 1999; 2009; Knight 2002, 149). But the question remains whether we should encourage students to become inventive consumers, experimenting end-users of technological amusements, or critical agents who are motivated to understand and willing to improve our complex techno-cultural environment. Martha Nussbaum supports intensive improvement of scientific and technical education, reminding us that capabilities like critical thinking and the ability to transcend local loyalties by approaching global problems as a citizen of the world "are at risk of getting lost in the competitive flurry" (Nussbaum 2010, 7). Nussbaum's insight into the importance of developing science education in collaboration with the humanities and the arts is in line with a recent conception of Gayatri Spivak, who argues for an aesthetic education which is inseparable from an ethical education: one that prepares individuals and communities for a mindful and ethical use of cutting-edge technology (Spivak 2012). As recent changes in the US science research and education policy also reminds us, the conception of STEM (Science, Technology, Engineering and Mathematics) integration needs to be complemented with the arts and need to put a strong emphasis on humanities. The STEM has to take up Arts as well, and need to be changed to STEAM. [16]

In the STEAM spirit, we believe that to improve mathematical literacy and abilities in Serbia and elsewhere, it is important to research new art- and culture-related contents in mathematics education and to develop experience based learning through the arts approaches to mathematical knowledge in classrooms, leading to creative applications of mathematics using hands-on models, digital environments to present the cultural, interdisciplinary, and artistic embeddedness of mathematics. What is common to all the approaches discussed here is the playful attitude in the serious business of teaching and learning. When trying to address the widening gap between general mathematical competence and increasingly computational contemporary culture, with grave social implications, we need to be courageous enough to trust in the power of play as one of the most fundamental human traits. It may also be the key to provoke the digital natives to get engaged in the field of inquiry so deeply embedded in their daily amusements.

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[1] The Mathematics in Society Project (MISP) began in 1980 as an international association of mathematics educators. They realised there was a paradox in that mathematics is widely used and diffused implicitly in all industrialized societies, but most pupils find school mathematics difficult and/or unpleasant (Rogerson 1986).

[2] See the proceedings of the world largest mathematics and arts community, the Bridges Conferences in 15 volumes < http://bridgesmathart.org/past-conferences/ > (Retrieved on 18 December 2014); the numerous issues of the journal SYMMETRY: Culture and Science < http://symmetry.hu/aus_journal_content_abs.html > (Retrieved on 18.12.2014); Issues of The Journal of Mathematics and the Arts < http://www.tandfonline.com/loi/tmaa20 > (Retrieved on 18 December 2014), etc.

[3] No. 530394-TEMPUS-1-2012-1-HU-TEMPUS-JPHES. Project Leader: Eszterházy Károly College, Hungary. Project members: University of Jyväskylä (Finland), Sint-Lucas School of Architecture (Belgium), University of Applied Arts Vienna (Austria), Belgrade Metropolitan University (Serbia), University of Novi Sad (Serbia), Serbian Academy of Sciences and Arts (Serbia), ICT College of Vocational Studies (Serbia).

[4] See: < http://vismath.ektf.hu/ > (Retrieved on 18 December 2014.)

[5] See: < http://vismath.ektf.hu/index.php?l=en&m=233&ss=1 >(Retrieved on 18 December 2014.)

[6] See:< http://vismath.ektf.hu/index.php?l=en&m=233 > (Retrieved on 18 December 2014.)

[7] Rural population (% of total population) in Serbia according to the World Bank: < http://www.tradingeconomics.com/serbia/rural-population-percent-of-total-population-wb-data.html >(Retrieved on 18 December 2014.)

[8] Due to the large number of participants all correlations are significant and relatively low correlations or medium strength correlations could be considered as interesting and important indicators of results.

[9] Cf. Elster & Ward, 2007.

[10] See the program's website: www.experienceworkshop.hu (Retrieved on 18 December 2014.)

[11] See: < http://vismath.ektf.hu/index.php?l=en&m=233&ss=1 > (Retrieved on 18 December 2014.)

[12] See: < http://vismath.ektf.hu/index.php?l=en&m=233 > (Retrieved on 18 December 2014.)

[13] In the form of lesson plans, the results of the exchange between EWM specialists and summer school participants are available at the Tempus project homepage and ready for dissemination in wider circles of Serbian mathematics teachers and so as to introduce them into the Serbian teachers' education:< http://vismath.ektf.hu/index.php?l=en&m=311 > (Retrieved on 18 December 2014.)

[14] This chapter is partially based on Darvas & Fenyvesi, 2014.

[15] EWM's International Traveling Exhibition is a constantly growing collection of artworks and mathematics modelling tools and math-art puzzle sets, with nearly 150 pieces by artists and scholars from all over the world. These artworks are key pieces in EWM's events. They can be employed to illustrate the cultural, artistic, architectural and interdisciplinary connections of mathematical thinking in many different ways. By the EWM's initiative, the Eszterházy Károly College of Eger, Hungary, set up a gallery at its' campus, which has been operating as an experimental math-art-education gallery and workshop space since 2011. This gallery, whose unique themes and concept are reflected in its name and in its slogan — Ars GEometrica Gallery: Interactions and Border-Crossings in Art and Science — functions as a completely new scene in the Hungarian mathematics teacher education: < www.arsgeo.hu/en/ > (Retrieved on 18 December 2014.)

[16] Cf. Strategies for Arts + Science + Technology Research: Executive Report on a Joint Meeting of the National Science Foundation and the National Endowment for the Arts:< http://cms.mit.edu/news/Harrell-NSF-NEA-Workshop-ExecutiveReportFinalDraft.pdf >(Retrieved on 18 December 2014.)

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